Solution of the differential equations

1. 1st order classical differential equations

$$a_1 \frac{dx}{dt} + a_0 x = b_0 u$$

Chapter 2

Models for Dynamic Systems & system similarity

as $x : x(t)$ response variable

$u : u(t)$ external inputs

a_0, a_1, b_0 Constants

$x(0) = x_0$ & $\dot{x}(0) = \dot{x}_0$

$$a_1 D x + a_0 x = b_0 u$$

$$\left[\frac{a_1}{a_0} D + 1\right] x = \frac{b_0}{a_0} u$$

$$\boxed{[\underbrace{\tau}_{\text{time Constant}} D + 1] x = \underbrace{G}_{\text{Gain}} u}$$

$\tau \dot{x} + x = G u$

time Constant

It's a measure of the speed of the response of a 1st order systems. (units of time)

Gain

$G = \frac{b_0}{a_0}$

* The solution has the form of $\boxed{x = x_h + x_p}$

as x_h : homogenous solution

(instantenous ~)

x_p : particular ~

(steady state ~)

1.1. Free response $u = 0$ $\quad X = X_h + \cancel{X_p} \rightarrow 0$

$$\boxed{\tau \dot{X} + X = 0} \qquad \therefore \boxed{\dot{X} = \frac{-1}{\tau} X}$$

The homogeneous solution is of the form $X = A e^{\lambda t}$

$\therefore \dot{X} = A \lambda e^{\lambda t}$

sub. in the 1st order eq^{n}

$$\cancel{A} \lambda \cancel{e^{\lambda t}} = \frac{-1}{\tau} \cancel{A e^{\lambda t}}$$

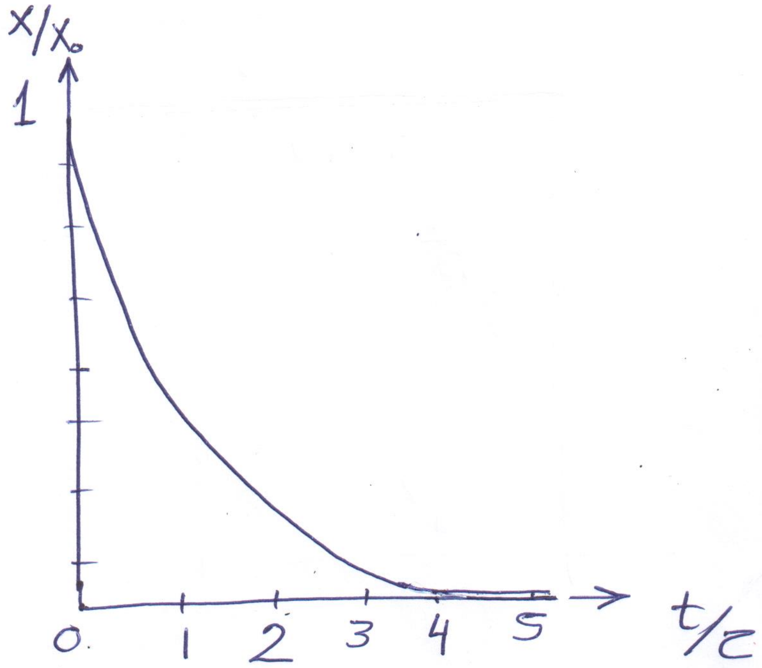
$$\therefore \boxed{\lambda = \frac{-1}{\tau}} \qquad \therefore X(t) = A e^{\frac{-t}{\tau}}$$

Applying the initial condition to find the constant A $\quad X(0) = X_0 = A \cancel{e^{0}} \nearrow 1$

$\therefore A = X_0$

the complete solution will be :-

$$\boxed{X(t) = X_0 \, e^{\frac{-t}{\tau}}}$$

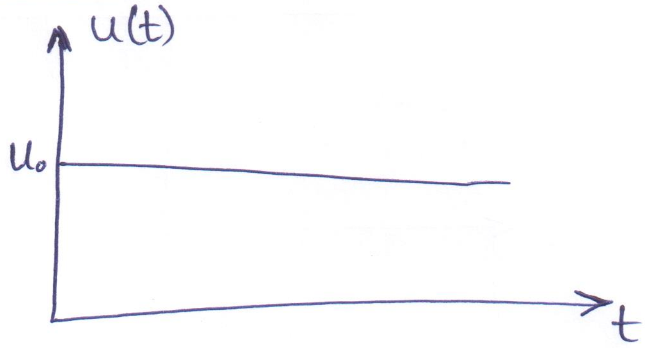


1st order systems will be very near to their final response after a time equal to $\underline{4\tau}$ has collapsed.

$$\boxed{\begin{aligned} X &= X_h + X_p \\ X &= A e^{\lambda t} + 0 \end{aligned}}$$

1.2. step input response $u(t) = U_o$

$$\boxed{\tau \dot{x} + x = G U_o}$$



as

$$x(t) = x_h(t) + x_p(t)$$

$$x_h = A e^{-t/\tau} \quad \& \quad x_p = B$$

$$\dot{x}_p = 0 \qquad \text{as } B \text{ is Constants}$$

* to find B, solve 1st order differential eqn for x_p only.

$$\tau \dot{x}_p + x_p = G U_o$$

$$0 + B = G U_o \qquad \therefore \boxed{B = G U_o}$$

* $x(t) = A e^{-t/\tau} + B$

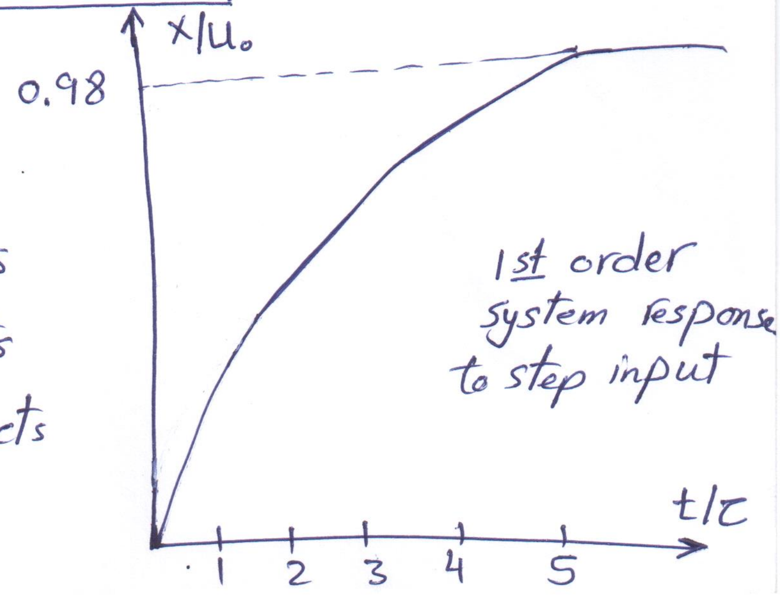
to find the Constant A apply the initial Conditions.

$$x_o = A \cancel{e^0}^{\,1} + G U_o \qquad \boxed{\therefore A = x_o - G U_o}$$

$$\therefore x(t) = [x_o - G U_o]\, e^{-t/\tau} + G U_o$$

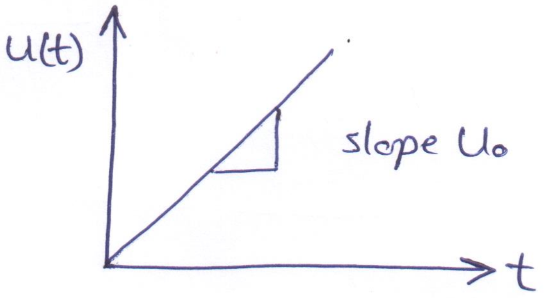
$$\boxed{x(t) = x_o e^{-t/\tau} + G U_o [1 - e^{-t/\tau}]}$$

* As time increases (after a time equal four times from τ), $x(t)$ approaches τU_o and all dynamic effects died away (all derivative effects are negligible).



1st order system response to step input

1.3. Ramp input response $u(t) = U_o t$



$$\boxed{\tau \dot{X} + X = G U_o t}$$

as $X(t) = X_h(t) + X_p(t)$

$X_h = A e^{-t/\tau}$ & $X_p = Rt + Q$

$\dot{X}_p = R$ as Q & R are constants

/* to find R & Q, solve 1st order differential eqn for X_p only

$$\tau \dot{X}_p + X_p = G U_o t$$

$$\tau R + (Rt + Q) = G U_o t$$

$$\underline{\tau R + Q} + Rt = G U_o t$$

at $t=0$ $\boxed{R = G U_o}$ as $\tau R + Q = 0$

$$\tau R + Q = 0$$

$$Q = -\tau R \Rightarrow \boxed{Q = -\tau G U_o}$$

$\therefore X_p = Rt + Q = G U_o t - \tau G U_o = G U_o (t - \tau)$

/* to find the Constant A apply the initial Conditions

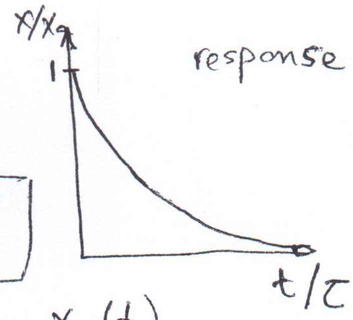
$$X_o = A \cancel{e^{0}}^{\,1} + \cancel{G U_o t}^{\,0} - \tau G U_o$$

$X_o = A - \tau G U_o$ $\therefore \boxed{A = X_o + \tau G U_o}$

$$\therefore X(t) = X_o e^{-t/\tau} + \tau G U_o e^{-t/\tau} + G U_o t - \tau G U_o$$

$$\boxed{X(t) = X_o e^{-t/\tau} + G U_o \left[t - \tau \left(1 - e^{-t/\tau}\right) \right]}$$

$$\boxed{(\tau D + 1)\, x = G u}$$



(*) If $u = 0$ free response

$\hookrightarrow u(t)$

$x_h = A e^{-t/\tau}$, $x_p = 0$

$\rightarrow$ $x(t) = A e^{\lambda t}$

$$\boxed{x(t) = x_0 e^{-t/\tau}}$$

$x(t) = x_h(t) + x_p(t)$

(*) If $u = u_0$

step input response

$x_h = A e^{-t/\tau}$, $x_p = \text{Const.} = B$

$\rightarrow$ $x(t) = x_0 e^{-t/\tau} + G u_0 \left[1 - e^{-t/\tau}\right]$

(*) If $u = u_0 t$

ramp input response

$x_h = A e^{-t/\tau}$, $x_p = Q + Rt$

$\rightarrow$ $x(t) = x_0 e^{-t/\tau} + G u_0 \left[t - \tau\left(1 - e^{-t/\tau}\right)\right]$

(*) If $u = u_0 \cos \omega t$

sinusoidal input response.

$x_h = A e^{-t/\tau}$, $x_p = B \cos(\omega t + \phi)$

$\rightarrow$ $x(t) = x_0 e^{-t/\tau} + G u_0 *$

$$\left[\frac{\cos(\omega t + \phi)}{\sqrt{1 + \omega^2 \tau^2}} - \frac{e^{-t/\tau}}{\sqrt{1 + \omega^2 \tau^2}}\right]$$

$x = x_h + x_p$

x_h: homogeneous solution (instantenous solution)

x_p: particular soln (steady state soln)

1.4. Sinusoidal Input response

$$\boxed{\tau \dot{x} + x = G U_o \cos \omega t}$$

as $x(t) = x_h(t) + x_p(t)$

$x_h = A e^{-t/\tau}$ & $x_p = B \cos(\omega t + \phi)$ as B is Const.

* to find B solve 1st order diff. eqn for x_p only

$$\boxed{B = \frac{G U_o}{\sqrt{1+\omega^2 \tau^2}}}$$

* to find A apply the I.C.

$$\boxed{A = x_o - \frac{G U_o}{[1+\omega^2 \tau^2]}}$$

$$\therefore \boxed{x(t) = x_o e^{-t/\tau} + G U_o \left[\frac{\cos(\omega t + \phi)}{\sqrt{1+\omega^2 \tau^2}} - \frac{e^{-t/\tau}}{(1+\omega^2 \tau^2)} \right]}$$

2. 2nd order differential equation

$$a_2 \frac{d^2 x}{dt^2} + a_1 \frac{dx}{dt} + a_o x = b_o U$$

$$a_2 D^2 x + a_1 D x + a_o x = b_o U$$

$$\left(\frac{a_2}{a_o} D^2 + \frac{a_1}{a_o} D + 1 \right) x = \frac{b_o}{a_o} U$$

$$\boxed{\left(\frac{D^2}{\omega_n^2} + 2\zeta \frac{D}{\omega_n} + 1 \right) x = G U}$$

ω_n : natural frequency $\omega_n = \sqrt{\frac{a_o}{a_2}}$

ζ : damping ratio $\frac{2\zeta}{\omega_n} = \frac{a_1}{a_o}$ $\therefore \zeta = \frac{a_1}{2\sqrt{a_o a_2}}$

G : static gain $G = \frac{b_o}{a_o}$

ξ : is the ratio of the value of damping in the system to the value of damping which defines the boundary between oscillatory & non-oscillatory behaviour.

2.2. Free response

$$\frac{\ddot{x}}{\omega_n^2} + \frac{2\xi}{\omega_n}\dot{x} + x = 0$$

with $x(0) = x_o$ & $\dot{x}(0) = \dot{x}_o$

Assume $x(t) = c\,e^{\lambda t}$

$$c\frac{\lambda^2}{\omega_n^2}e^{\lambda t} + c\frac{2\xi}{\omega_n}\lambda e^{\lambda t} + c e^{\lambda t} = 0$$

$\boxed{\div\, c e^{\lambda t}}$ as $c \neq 0$ & $e^{\lambda t} \neq 0$

$$\therefore \frac{\lambda^2}{\omega_n^2} + \frac{2\xi}{\omega_n}\lambda + 1 = 0$$

$$\therefore \lambda_{1,2} = \omega_n\left(-\xi \pm \sqrt{\xi^2 - 1}\right) \longrightarrow \#$$

2.2.1. if $\xi < 1$ $\therefore$ under-damped case

$$\lambda_{1,2} = \omega_n\left(-\xi \pm i\sqrt{1-\xi^2}\right)$$

as $i = \sqrt{-1}$

as $\omega_d = \omega_n\sqrt{1-\xi^2}$

$$= -\xi\omega_n \pm i\,\omega_d$$

$$\therefore x(t) = C_1 e^{(-\xi\omega_n + i\omega_d)t} + C_2 e^{(-\xi\omega_n - i\omega_d)t}$$

$$= e^{-\xi\omega_n t}\left(C_1 e^{i\omega_d t} + C_2 e^{-i\omega_d t}\right)$$

$$= e^{-\xi\omega_n t}\left(A\sin\omega_d t + B\cos\omega_d t\right)$$

General solution

$$X(t) = X_o e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right] + \frac{\dot{X}_o}{\omega_d} e^{-\zeta\omega_n t}\sin\omega_d t$$

2.2.2. $\zeta > 1$ ∴ overdamped Case

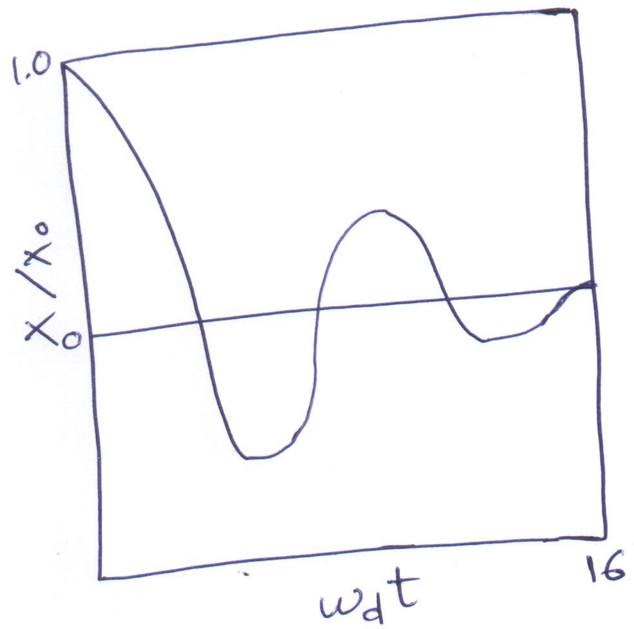
λ_1 & λ_2 are real

$$X(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

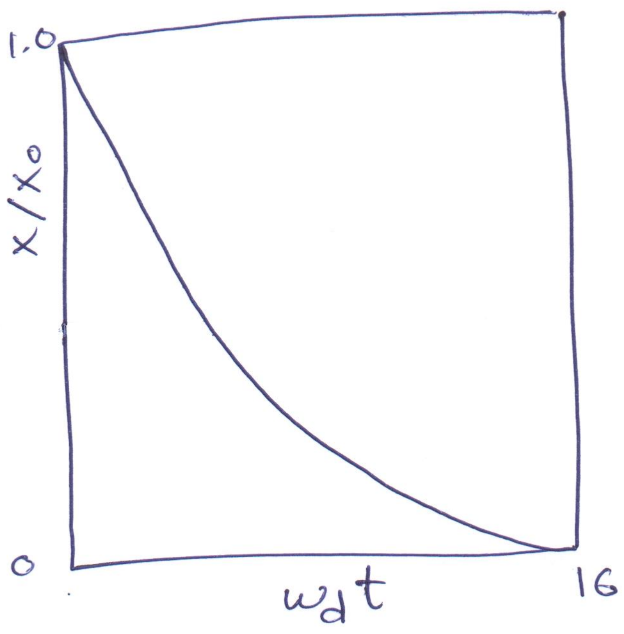
as C_1 & C_2 are constants
λ_1 & λ_2 are negative (disp. decreases with time)

2.2.3. $\zeta = 1$ ∴ critical damping Case

use the underdamped solution with $\zeta = 0.9999$
or " " overdamped " " $\zeta = 1.0001$



Free response of underdamped 2nd order system

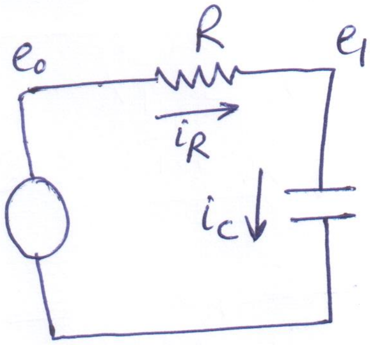


overdamped

* state space format for differential equations

It Consists of a set of simultaneous 1st order diff. eqns

Ex.



$e_o(t)$: input voltage
$e(t)$: output voltage

state space eqns

$$e_o - e_1 = R\, i_R \longrightarrow (a)$$
$$i_c = C\dot{e}_1 \longrightarrow (b)$$
$$i_R = i_c \longrightarrow (c)$$

electrical system

single 1st order differential equations

$$\dot{e}_1 = \frac{-1}{RC} e_1 + \frac{1}{RC} e_o$$

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from eqn (a) $\quad i_R = \frac{1}{R}(e_o - e_1)$

from eqn (b) $\quad \dot{e}_1 = \frac{1}{C}\, i_c$

from eqn (c) in the previous eqn

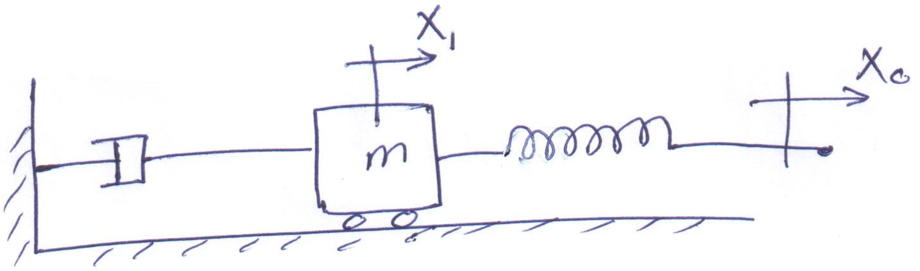
$$\dot{e}_1 = \frac{1}{C}\, i_R$$
$$= \frac{1}{C}\left(\frac{1}{R}(e_o - e_1)\right)$$
$$= \frac{1}{RC} e_o - \frac{1}{RC} e_1$$

$$\dot{e}_1 = \frac{-1}{RC} e_1 + \frac{1}{RC} e_o$$

state variable

input

Ex.



motion of mass discribed by x_1

〃 〃 free end 〃 〃 x_0

2nd order modelling eqn

$$m\ddot{x}_1 + b\dot{x}_1 + Kx_1 = Kx_0 \longrightarrow (a)$$

state space eqns

* state variables are x_1 , v_1

& $\dot{x}_1 = v_1$ & $\ddot{x}_1 = \dot{v}_1$

from eqn (a)

$$m\dot{v}_1 + bv_1 + Kx_1 = Kx_0$$

$$\boxed{\begin{aligned} \dot{v}_1 &= \frac{-b}{m}v_1 - \frac{K}{m}x_1 + \frac{K}{m}x_0 \\ \& \quad \dot{x}_1 &= v_1 \end{aligned}}$$

state space eqns

$x_0(t)$ is the input to the set

* Engineering system similarity

- Many physical different Kinds of dynamic systems have the same or <u>similar differential equations</u> and have <u>similar response</u> behaviors.
- Dynamic systems in different disciplines Contain the <u>same basic elements</u> (resistance, capacitance and inductance).

- <u>Effort</u>: is the potential <u>or</u> ability to do work.
- <u>Effort variable</u>: is the system variable expresses the effort
- <u>Flow</u>: is the rate at which work can be done with time.
- <u>Flow variable</u>: is the system variable that expresses the flow <u>or</u> rate of change with time.

$$\boxed{\text{effort} = \text{resistance} * \text{flow}}$$

* <u>Impedance</u>: is the ratio of its effort variable to its flow variables

$$\boxed{\text{Impedance} = \frac{\text{effort}}{\text{flow}}}$$

<u>static</u>

resistance

<u>Dynamic</u>

* Capacitance
* inductance

* Dissipative elements :

They are the elements that dissipate energy or Convert it to another form.

* Effort storage elements: (Capacitors) [capacitive elements]

They store energy by virtue of the flow variable.

$$\boxed{\text{flow} = \text{Capacitive} * \frac{d}{dt}\,\text{effort}}$$

* flow storage elements: [inductive elements]

They store energy by virtue of the flow variable

$$\boxed{\text{effort} = \text{inductance} * \frac{d}{dt}\,\text{flow}}$$

* For a mechanical system

- Effort is the force or torque
- flow is the velocity ($\frac{dx}{dt}$)
- Dissipative element is the dashpot or damper (viscous friction) & dry friction (columb friction)
- Capacitive element is the spring

$$\boxed{\begin{aligned} \text{Velocity} &= \frac{1}{\text{stiffness}} * \frac{d}{dt}\,\text{force} \\ \text{force} &= \text{stiffnes} * \text{position} \end{aligned}}$$

$V = \frac{dx}{dt}$

$x = \int v\,dt$

- Inductive element is the mass or inertia & the flywheel.

$$\boxed{\begin{aligned} \text{Force} &= \text{mass} * \frac{d}{dt}\,\text{velocity} \\ \text{torque} &= \text{inertia} * \frac{d}{dt}\,\text{angular velocity} \end{aligned}}$$

* Conservation laws of linear and angular momentum

linear

$$\Sigma F_{net} = \frac{d}{dt}(mV)$$ (Newton's 2nd law)

$$\Sigma F_{net} - m\frac{dv}{dt} = 0$$ (D'Alembert principle)

angular

$$\Sigma T_{net} = \frac{d}{dt}[J\omega]$$ (Newton)

$$\Sigma T_{net} - J\frac{d\omega}{dt} = 0$$ (D'Alembert)

* For an electrical system

- Effort is the voltage
- flow is the current.
- Dissipative element is the resistance.
- capacitive element is the electrical capacitor

$$\boxed{\text{Current} = \text{Capacitance} * \frac{d}{dt}\,\text{voltage}}$$

- inductive element is the inductor

$$\boxed{\text{effort} = \text{inductance} * \frac{d}{dt}\,\text{current}}$$

* Conservation laws of charge

$$\sum i_{node} = \frac{dQ}{dt} = c\,\frac{de}{dt}$$

$$\sum i_{node} - c\,\frac{de}{dt} = 0$$

- the charge in an electrical system is Constant
- sum of all Currents at a node is equal to the rate at which charge is being stored at the node.

* For a Fluid System

- Effort is the pressure
- flow is the volume flow rate
- Dissipative element is a capillary viscous resistor (pressure loss due to viscous shear) <u>or</u> an orifice (Bernoulli velocity head loss due to sudden expansion).

- Capacitive element is the accumulator <u>or</u> volume of compressible fluid in a tank

$$\boxed{\text{flow} = \text{Capacitance} * \frac{d}{dt}\,\text{pressure}}$$

$$\text{flow} = \frac{d}{dt}\,\text{volume}$$

- inductive element is a long line of small cross-sectional area.

$$\boxed{\text{pressure} = \text{inductance} * \frac{d}{dt}\,\text{flow rate}}$$

fluid inductance is caused by the inertial properties of a fluid as it accelerates in a pipe.

* Conservation law of mass

The mass of a system fluid is Constant

$$\Sigma \dot{m}_{net} = \frac{d}{dt}(\rho V) = \dot{\rho} V + \dot{V} \rho$$

<u>or</u> $$\Sigma \dot{m}_{net} - \frac{d}{dt}(\rho V) = 0$$

* For a thermal system

- Effort is the temperature
- flow is the heat flow
- Dissipative element is the resistance due to heat transfer through
  - Convection
  - Conduction
  - radiation
- capacitive element is the mass of element times the specific heat of the material.

$$\boxed{\text{heat flow} = \text{Capacitance} * \frac{d}{dt}\,\text{temperature}}$$

- inductive element
  - inductance does not exist.

* Conservation law of energy

$$\underbrace{\Sigma Q_h}_{\text{heat transfer}} - \underbrace{\dot{W}}_{\text{work}} + \underbrace{\dot{m}_{net}}_{\text{mass flow rate}}\Big[\underbrace{h}_{\text{specific enthalpy}} + \frac{V^2}{2} + z\Big] = \frac{d}{dt}\Big[\underbrace{m}_{\text{mass}}u + m\frac{V^2}{2} + mz\Big]_{\underbrace{C.V.}_{\text{Control Volume}}}$$

The energy in a system is Constant if there is no energy exchange with the environment and no energy is being stored, this equation reduces to Bernolli equation

$$\frac{P}{\rho g} + \frac{V^2}{2g} + z = \text{Constant along a stream line}$$