Example 3.5 Spring Scale

A schematic of a simple spring scale used to measure items that are sold by weight is shown in Figure 3.11. The weighing pan has a mass of m. Develop an equation describing the motion of the pan.

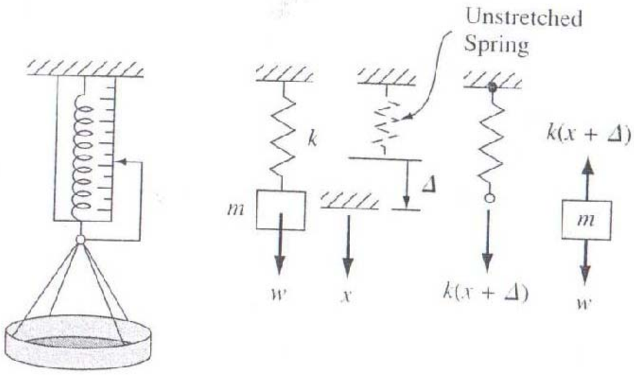

Figure 3.11 Spring scale.

Solution When the spring is unstretched, it exerts no force on the weighing pan. When the pan is installed, its weight stretches the spring an amount Δ until the spring exerts a force on the pan that is equal to the weight of the pan. That is,

$$k\Delta = W = mg \tag{3.36}$$

The quantity Δ determines the static equilibrium position of the system. If we use x to measure the displacement of the pan from this *static equilibrium position*, the forces acting on the pan are its weight W and the spring force $F_s = k(x + \Delta)$.

Applying Newton's second law to the mass, we obtain

$$\sum F_x = m\ddot{x} \tag{3.37}$$

or

$$W - F_s = m\ddot{x} \tag{3.38}$$

Thus, we have

$$W - k(x + \Delta) = m\ddot{x} \tag{3.39}$$

which reduces to

$$m\ddot{x} + kx = W - k\Delta = 0 \tag{3.40}$$

Hence, if the motion of a linear mechanical spring-mass-damper system acted on by gravity is measured from its equilibrium position, the equation of motion takes the simple form of Eq. (3.40), in which the weight forces are canceled by the initial spring forces due to the static deflection. Any subsequent motions are calculated with respect to the equilibrium position.

We have neglected energy dissipation in this model, and the equation we derived cannot predict any decay of the motion of the pan once it is disturbed from its equilibrium position. To simulate the decay, the model would have to be modified to include energy dissipation of some kind.

Example 3.6 Machine Part

Figure 3.12 shows a machine part that slides along a smooth lubricated surface that itself is attached to a fixed base by a spring. In its operation, the part is subjected to a force that varies harmonically with time at a frequency of f Hz = ω rad/s. Derive the governing equation of motion for the mass m_2.

Solution In Figure 3.13, the reference axis x is taken positive to the right, and the masses are labeled m_1 and m_2. The oil film is modeled as a viscous damping element. Free-body diagrams of the masses, spring, and damper are shown in Figure 3.14, with

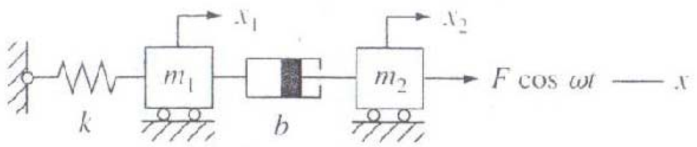

Figure 3.13 Machine part model.

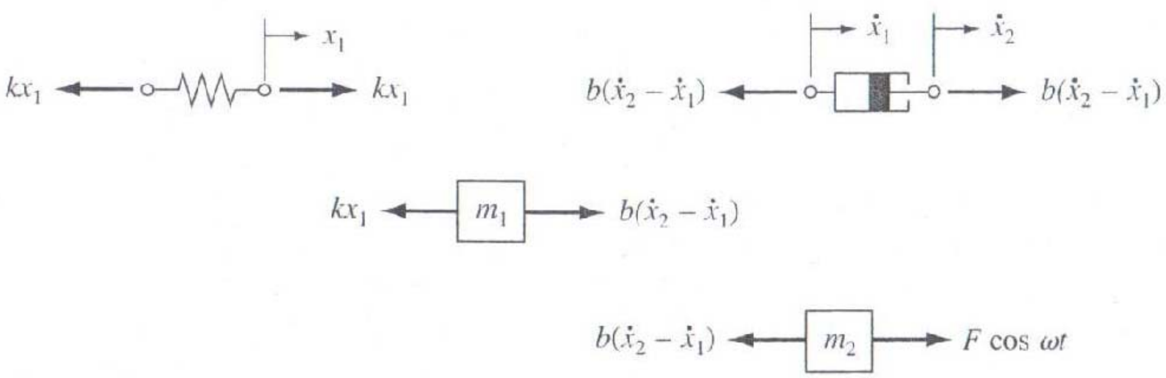

Figure 3.14 Free-body diagrams.

forces that are consistent with the positive displacements shown in the schematic of the system.

Newton's second law is now applied to each of the masses:

Node 1:

$$\sum F = m_1\ddot{x}_1 \tag{3.41}$$

$$-kx_1 + b(\dot{x}_2 - \dot{x}_1) = m_1\ddot{x}_1 \tag{3.42}$$

Node 2:

$$\sum F = m_2\ddot{x}_2 \tag{3.43}$$

$$-b(\dot{x}_2 - \dot{x}_1) + F\cos\omega t = m_2\ddot{x}_2 \tag{3.44}$$

Thus, the two second-order equations of motion become

$$m_1\ddot{x}_1 + b\dot{x}_1 - b\dot{x}_2 + kx_1 = 0 \tag{3.45}$$

$$m_2\ddot{x}_2 + b\dot{x}_2 - b\dot{x}_1 = F\cos\omega t \tag{3.46}$$

We may also write these equations in the following form:

$$[m_1D^2 + bD + k]x_1 - bDx_2 = 0 \tag{3.47}$$

$$[m_2D^2 + bD]x_2 - bDx_1 = F\cos\omega t \tag{3.48}$$

We solve for x_1 from the first equation and substitute into the second equation:

$$x_1 = \frac{bDx_2}{[m_1D^2 + bD + k]} \tag{3.49}$$

$$[m_2D^2 + bD]x_2 - \frac{b^2D^2x_2}{[m_1D^2 + bD + k]} = F\cos\omega t \tag{3.50}$$

After simplification, we obtain the following third-order equation:

$$[m_1m_2D^4 + (m_1 + m_2)bD^3 + m_2kD^2 + bkD]x_2 = [m_1D^2 + bD + k]F\cos\omega t \tag{3.51}$$

We next consider an example in which the input to the system is a displacement instead of a force.

Example 3.7 Vehicle Suspension System

A simplified translational model of an automotive suspension system is constructed by considering only the translational motion of one wheel of the vehicle (quarter-car model). The model is shown in Figure 3.15. The stiffness of the tire is modeled by a linear spring, the tire, axle, and moving parts by a mass m_1, the suspension system by a spring and viscous damper (shock absorber), and the supported vehicle components by a mass m_2.

Solution We follow the same basic steps as used in previous examples. A coordinate system for translation is selected, with y taken positive upwards. If the wheels stay in contact with the ground, the lower end of the tire spring follows the surface of the roadway, as described by $y_0(t)$. The springs, damper, and masses are separated from their attached components so that the forces acting on them can be identified in terms of the parameters of the problem. Figures 3.16 and 3.17 illustrate the appropriate free-body diagrams. We next use Newton's second law to write the equations of motion for each mass and imply equilibrium of forces at points where elements join.

We measure y_1 and y_2 from the at-rest equilibrium position of the springs, acted on by the weight of the vehicle's components. This means that the forces due to the weights of the components are balanced by the preload in the springs, and the two will cancel each other in the equations of motion. Thus, to simplify the process, we write the equations of motion without including the weight terms and understand that y_1 and y_2 are measured from the equilibrium position. (See Example 3.5.)

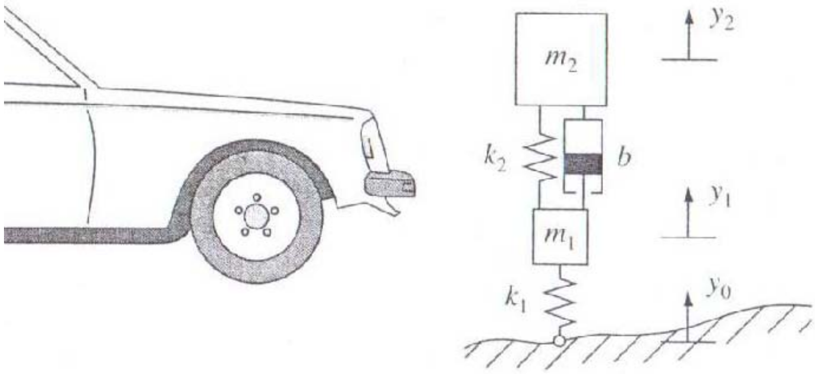

Figure 3.15 Auto front end and model.

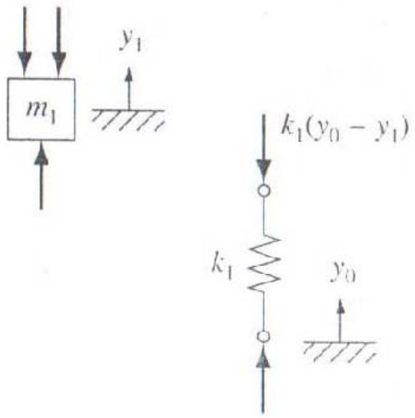

Figure 3.16 Free-body diagram of m_1.

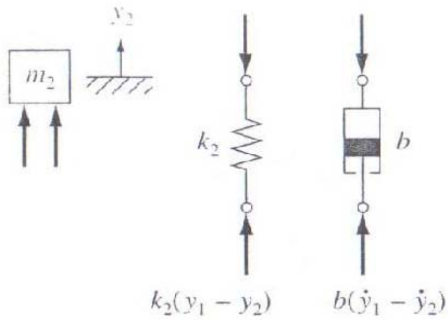

Figure 3.17 Free-body diagram of m_2

Recognizing that the springs will probably be in compression most of the time, we show the spring forces compressive and write the spring element force equations accordingly. In the development of an accurate set of governing equations, it doesn't matter whether elements are shown in tension or compression, as long as the element force equations are consistent with the convention adopted. Thus, for mass m_1,

$$\sum F_y = m_1 \ddot{y}_1 \tag{3.52}$$

$$k_1(y_0 - y_1) - k_2(y_1 - y_2) - b(\dot{y}_1 - \dot{y}_2) = m_1 \ddot{y}_1 \tag{3.53}$$

And mass m_2,

$$\sum F_y = m_2 \ddot{y}_2 \tag{3.54}$$

$$k_2(y_1 - y_2) + b(\dot{y}_1 - \dot{y}_2) = m_2 \ddot{y}_2 \tag{3.55}$$

The resulting equations describing the motion of the two masses are

$$m_1 \ddot{y}_1 + b(\dot{y}_1 - \dot{y}_2) + k_2(y_1 - y_2) + k_1 y_1 = k_1 y_0(t) \tag{3.56}$$

and

$$m_2 \ddot{y}_2 - k_2(y_1 - y_2) - b(\dot{y}_1 - \dot{y}_2) = 0 \tag{3.57}$$

Example 3.10 Engine and Propeller Model

A simplified model of a turboprop aircraft engine and propeller is shown in Figure 3.22. The mass moment of inertia of the rotating parts of the engine is represented by J_e, the mass moment of inertia of the propeller by J_p. The driving torque applied to the engine is $T(t)$. The drive shaft has a small mass moment of inertia in comparison to that of the engine and propeller and is represented by an inertialess discrete torsional spring. The rotation of the propeller is opposed by aerodynamic drag torque, which is proportional to the square of the rotational speed of the propeller. Develop a mathematical model for this system, and write its equations of motion.

Solution A reference axis x is established along the drive shaft as shown, and rotations θ_1 and θ_2 of inertias J_e and J_p are taken to be positive along this axis, according to the right-hand rule at the propeller end of the shaft. If it is assumed that $\theta_2 > \theta_1$, then

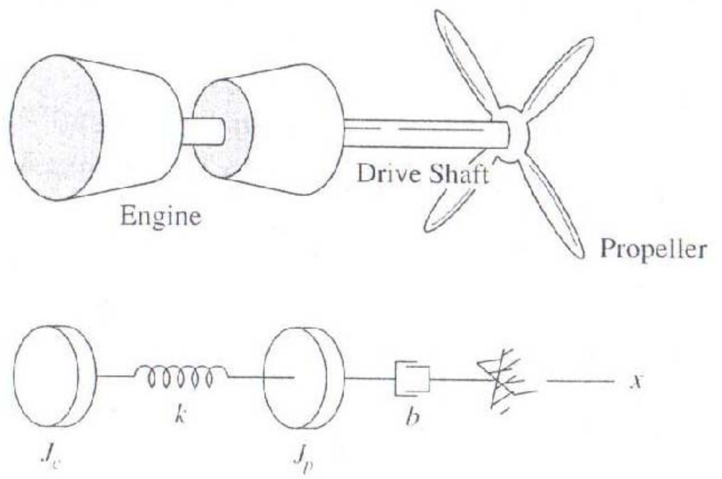

Figure 3.22 Engine-and-propeller model.

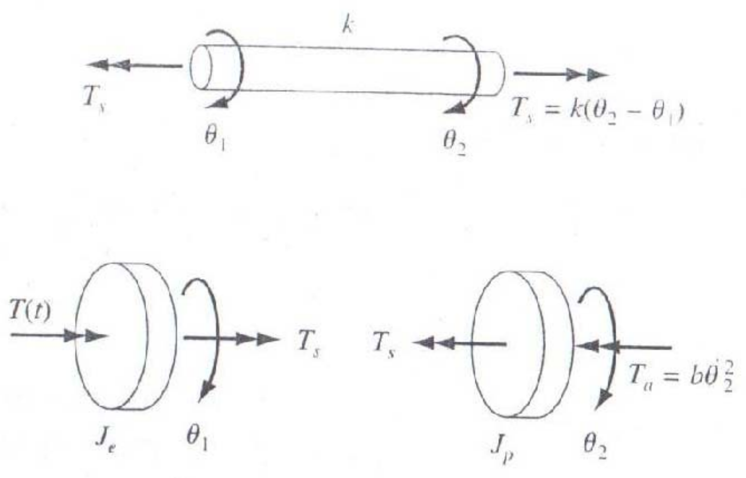

Figure 3.23 Engine and propeller free-body diagrams.

the drive shaft torques must be in the directions shown in Figure 3.23. Of course, the equations can just as easily and correctly be derived by taking $\theta_2 < \theta_1$ and reversing the torque; just be consistent.

The equations governing the motion of the two inertias become the following:
At the θ_1 node:

$$\sum M_x = J_e\ddot{\theta}_1 \tag{3.74}$$

$$T(t) + k(\theta_2 - \theta_1) = J_e\ddot{\theta}_1 \tag{3.75}$$

At the θ_2 node:

$$\sum M_x = J_p\ddot{\theta}_2 \tag{3.76}$$

$$-k(\theta_2 - \theta_1) - b\dot{\theta}_2^2 \text{ sign } (\dot{\theta}_2) = J_p\ddot{\theta}_2 \tag{3.77}$$

Equations (3.75) and (3.77) constitute the fourth-order model of the torsional system.

Example 3.11 Geared System

The shaft of Example 3.8 is fixed at one end and has the larger gear of a pair of gears at the other end. The pitch radii of the steel gears are $r_1 = 8$ inches (64 teeth) and $r_2 = 4$ inches (32 teeth); the tooth face widths are 0.5 inch. Find the equation of motion of this system if the smaller gear has a torque $T \sin \omega t$ applied to it, where ω is the frequency, in rad/s, of the excitation torque.

Solution The shaft and gears are sketched in Figure 3.24, with angular variables assigned to each gear. The angular coordinates are taken in opposite senses to account for the opposite rotations of two mating gears. The contact force between the mating gear teeth is called F.

The equations of motion for the two gears are:
For gear 1:

$$\sum M_x = J_1\ddot{\theta}_1 \tag{3.78}$$

$$-T_s + Fr_1 = J_1\ddot{\theta}_1 \tag{3.79}$$

For gear 2:

$$\sum M_x = J_2\ddot{\theta}_2 \tag{3.80}$$

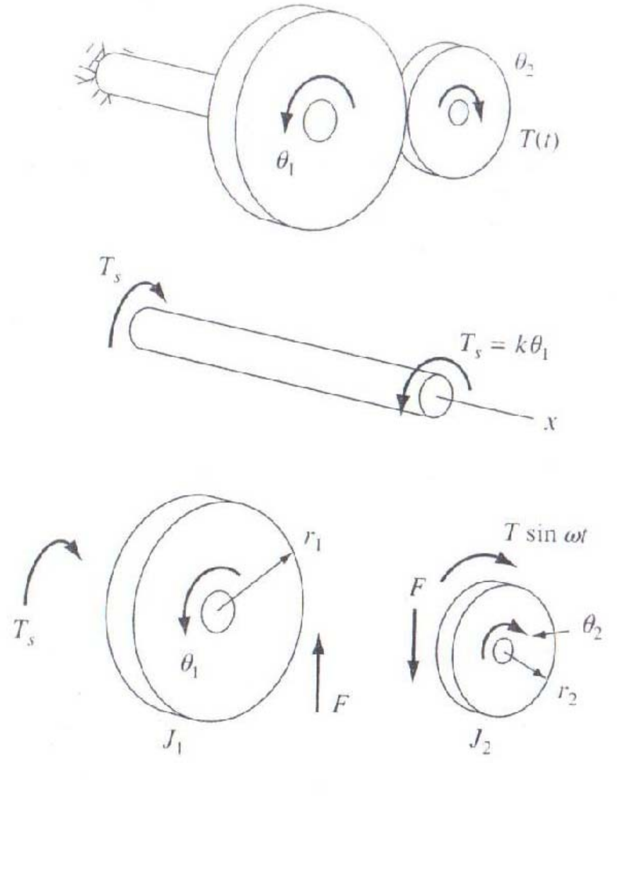

Figure 3.24 Shaft and gear system.

$$T \sin \omega t - F r_2 = J_2 \ddot{\theta}_2 \tag{3.81}$$

Solving for F from the last equation and substituting into the equation for gear 1, we obtain

$$F = -\frac{J_2 \ddot{\theta}_2 - T \sin \omega t}{r_2} \tag{3.82}$$

$$J_1 \ddot{\theta}_1 + k\theta_1 = F r_1 = -\frac{r_1}{r_2}(J_2 \ddot{\theta}_2 - T \sin \omega t) \tag{3.83}$$

$$J_1 \ddot{\theta}_1 + J_2 \ddot{\theta}_2 \frac{r_1}{r_2} + k\theta_1 = \frac{r_1}{r_2}(T \sin \omega t) \tag{3.84}$$

This problem requires only one variable to define the position of the gears. As the two gears turn in contact with each other, the arc lengths they traverse are equal. We have the following constraint equation to consider:

$$r_1 \theta_1 = r_2 \theta_2 \tag{3.85}$$

Thus, there is only one independent coordinate for the problem. We solve for θ_2 and differentiate the result to get $\ddot{\theta}_2$:

$$\theta_2 = \frac{r_1}{r_2}\theta_1 \quad \text{and} \quad \ddot{\theta}_2 = \frac{r_1}{r_2}\ddot{\theta}_1 \tag{3.86}$$

We substitute this result into Eq. (3.84) and obtain an equation in θ_1 only:

$$J_1 \ddot{\theta}_1 + J_2 \ddot{\theta}_1 \left(\frac{r_1}{r_2}\right)^2 + k\theta_1 = \frac{r_1}{r_2}(T \sin \omega t) \tag{3.87}$$

SYSTEMS OF COMBINED TRANSLATIONAL AND ROTATIONAL ELEMENTS

Mechanical systems are very often composed of combinations of translational and rotational elements rather than elements of just one type. However, the fundamental method of developing correct models for these systems remains the same as that used in previous examples. You should:

(1) Establish an inertial coordinate system (one that is not attached to an accelerating object).

(2) Identify and isolate the discrete system elements (springs, dampers, masses, rotational inertias).

(3) Determine the minimum number of variables needed to uniquely define the configuration of the system. This can be done by subtracting the number of constraints from the number of equations of motion.

(4) Establish convenient axis systems and make appropriate free body sketches, showing all variables and all forces and torques acting on the elements.

(5) For stiffness and damping elements, write the equations that relate element loadings to element deformation variables.

(6) Apply Newton's second law of motion at all nodes of the model.

(7) Combine equations as necessary to isolate response variables of interest.

These steps are essential for accurate modeling and simulation. The degree to which you accomplish them correctly, solve the resulting equations, and intelligently interpret the solution will determine your success in modeling and simulation. The next four examples illustrate the ideas we have discussed.

Example 3.12 Rolling Wheel

A portion of a mechanical device may be idealized as a uniform, homogeneous wheel rolling without slipping on a horizontal surface, as shown in Figure 3.25. The center of the wheel is fastened to the frame of the device by a linear spring, and a force is applied at the top of the wheel. Find the equation of motion that governs the horizontal position of the center of the wheel.

Solution In Figure 3.26, the wheel is shown in a displaced position, and the forces acting on it are indicated in a free-body diagram. Since the wheel rolls without slipping, a

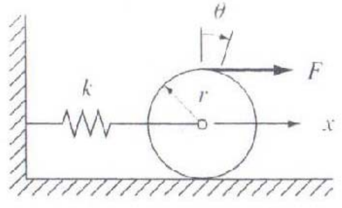

Figure 3.25 Rolling wheel and spring.

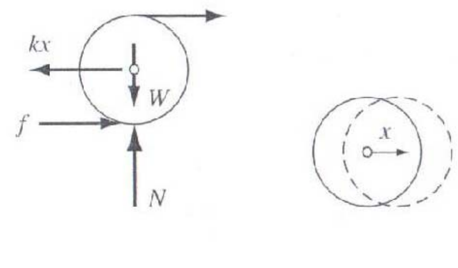

Figure 3.26 Free-body diagram of wheel.

frictional force f occurs at the point of contact between the wheel and the surface. Only one variable is required to uniquely define the location of the wheel; thus, the wheel displacement x and the wheel angular rotation θ are not independent. In our solution, we will use x. The wheel is not constrained to rotate about a fixed axis, so we use the general form of Newton's second law that includes x- and y-translations and refer the moment and mass moment of inertia to an axis through the center of gravity of the object:

$$\sum F_x = m\ddot{x}_{cg} \tag{3.89}$$

$$\sum F_y = m\ddot{y}_{cg} \tag{3.90}$$

$$\sum M_{cg} = J_{cg}\ddot{\theta} \tag{3.91}$$

The first and third equations may be used to develop a description of the horizontal motion of the wheel. Because the wheel rolls on a horizontal surface, there is no acceleration in the y direction, and the second equation expresses the equilibrium between the weight of the wheel and the normal force N at the surface. The first and third equations give

$$\sum F_x = m\ddot{x}_{cg} \qquad\qquad F + f - kx = m\ddot{x} \tag{3.92}$$

$$\sum M_{cg} = J_{cg}\ddot{\theta} \qquad\qquad rF - rf = J_{cg}\ddot{\theta} \tag{3.93}$$

We now solve for f from the moment equation:

$$f = F - J_{cg}\frac{\ddot{\theta}}{r} \tag{3.94}$$

Before substituting the right-hand side of Eq. (3.94) into the translation equation, we find θ in terms of x. Note that as the wheel rolls through an angle θ, the arc length $r\theta$ along the rim of the wheel is equal to the distance x traveled by the center. If it is helpful, think of the wheel on a stationary axis winding up a rope, the analog of the horizontal surface. The amount of rope wound onto the drum is $r\theta$. Thus,

$$x = r\theta \quad \text{and} \quad \ddot{x} = r\ddot{\theta}, \quad \text{so} \quad \ddot{\theta} = \frac{\ddot{x}}{r} \tag{3.95}$$

$$f = F - J_{cg}\frac{\ddot{x}}{r^2} \tag{3.96}$$

Substitution gives

$$F + \left(F - J_{cg}\frac{\ddot{x}}{r^2}\right) - kx = m\ddot{x} \tag{3.97}$$

$$m\ddot{x} + J_{cg}\frac{\ddot{x}}{r^2} + kx = 2F \tag{3.98}$$

$$\left(m + \frac{J_{cg}}{r^2}\right)\ddot{x} + kx = 2F \tag{3.99}$$

For a uniform circular disk, the mass moment of inertia with respect to an axis through its center of gravity is $mr^2/2$. Substituting this for J_{cg} gives

$$\frac{3}{2}m\ddot{x} + kx = 2F \tag{3.100}$$

If a rigid body rotates about a fixed axis O, the moment equation in Newton's second law can be written for that axis. Both the sum of the moments and the mass moment of inertia are then written for that axis:

$$\sum M_O = J_O\ddot{\theta} \tag{3.101}$$

Example 3.13 Trailing Arm Suspension System

A simplified model of an automotive suspension system is shown in Figure 3.27. The wheel is supported relative to the chassis by a torsion bar spring and a shock absorber. The tire stiffness is to be represented in the model. Develop the equations governing the angular motion of the pivot arm.

Solution An x-, y-, z-axis system is established as shown in Figure 3.28. The torsion bar and tire stiffnesses are represented by linear springs, the shock absorber by a viscous damper. The torsion arm is shown in a displaced position. It is assumed that

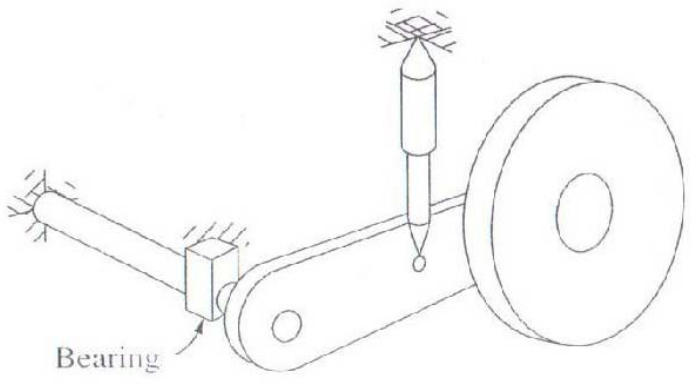

Figure 3.27 Automotive suspension system.

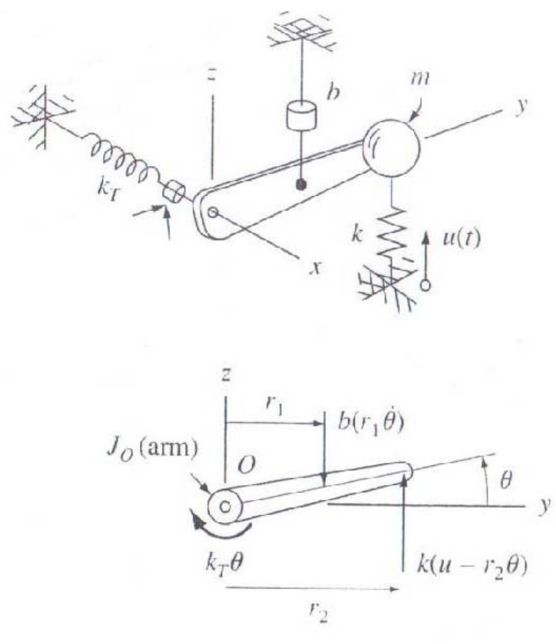

Figure 3.28 Suspension schematic and free-body diagram.

the angular motion is small enough that forces from the damper and tire spring remain essentially vertical. That assumption will need to be verified when the equations are solved to find the deformation. The pivot arm is assumed to be rigid, and the quantity $u(t)$ represents the variation in the height of the road surface and acts as an input to the system as the car moves along the road.

The tire spring, torsion spring, and shock absorber forces are

$$k(u(t) - r_2\theta), \quad k_T\theta, \quad \text{and} \quad b(r_1\dot{\theta}) \tag{3.102}$$

We use the moment equation form of Newton's second law, written for rotation about a fixed axis O:

$$\sum M_O = J_O\ddot{\theta} \tag{3.103}$$

$$-k_T\theta - r_1 b(r_1\dot{\theta}) + r_2 k(u(t) - r_2\theta) = J_O\ddot{\theta} \tag{3.104}$$

Here the total rotational inertia about the torsion bar axis is the inertia of the arm plus the inertia of the mass. The parallel axis theorem is used to calculate the total inertia. (see Appendix B):

$$J_O = J_{O(arm)} + J_{mass} \tag{3.105}$$

$$J_O = J_{O\,(arm)} + mr_2^2 \tag{3.106}$$

$$J_O\ddot{\theta} + r_1^2 b\dot{\theta} + (k_T + r_2^2 k)\theta = r_2 k\, u(t) \tag{3.107}$$

The quantity $u(t)$ is assumed to be a known function of time as the vehicle moves forward.

As an additional example of modeling mechanical systems we consider a hoisting system that contains a translational mass, a damper, and a spring, as well as a torsional spring and two rotational inertias.

Example 3.14 Hoisting System

Find the differential equations describing the motion of the hoisting system shown in Figure 3.29. A torque supplied by the motor at the right end of the shaft raises or lowers the mass m. The mass is guided so that it can move only in the vertical direction, and a viscous friction device between the container and its guides is used to damp out possible oscillations.

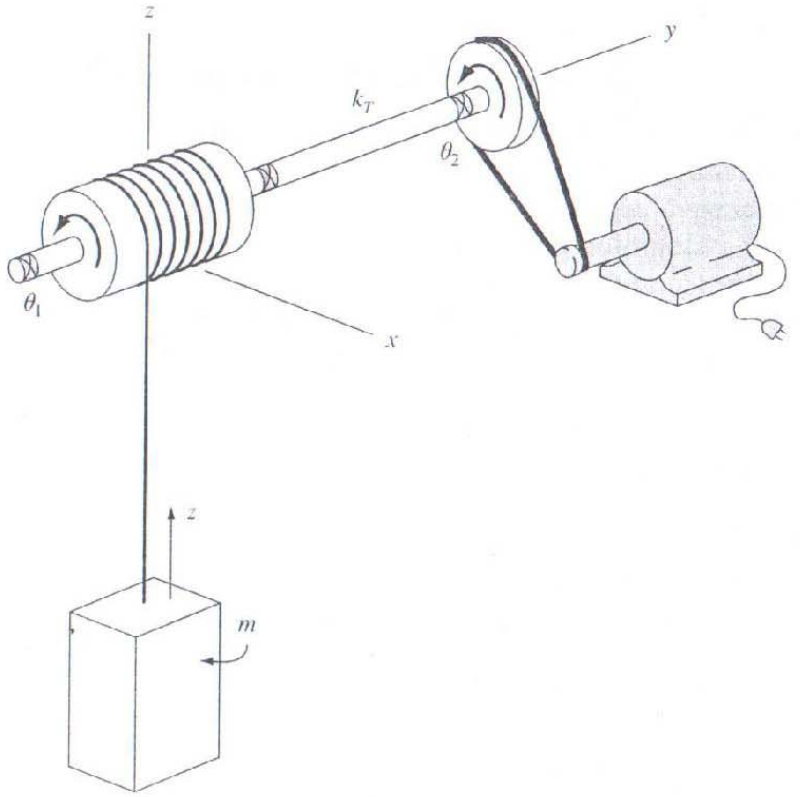

Figure 3.29 Hoisting system.

Solution First, we sketch the two spring elements and the damper element and show the loadings corresponding to positive values of the displacement variables. Next, letting k be the tensile stiffness of the hoisting cable, we write the element load-deformation equations in terms of these variables. Then we show the two inertias and the mass with compatible loadings acting on them. The element loadings are sketched in Figure 3.30. The element loading-deformation relations are

$$T_s = k_T(\theta_1 - \theta_2) \tag{3.108}$$

$$F = k(r_1\theta_1 - z) \qquad F_d = b\dot{z} \tag{3.109}$$

Applying Newton's second law to the two inertias and to the mass, we obtain the following equations:

At θ_1 node:

$$\sum M_y = J_1\ddot{\theta}_1 \tag{3.110}$$

$$-T_s - r_1F = J_1\ddot{\theta}_1 \tag{3.111}$$

$$J_1\ddot{\theta}_1 = -k_T(\theta_1 - \theta_2) - r_1k(r_1\theta_1 - z) \tag{3.112}$$

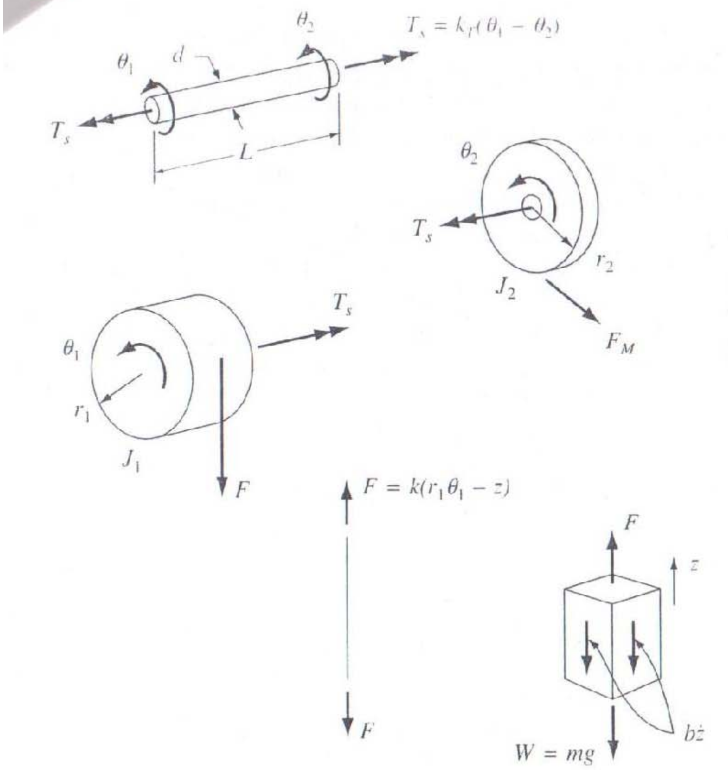

Figure 3.30 Hoisting system free-body diagrams.

At θ_2 node:

$$\sum M_y = J_2\ddot{\theta}_2 \tag{3.113}$$

$$T_s + r_2F_M = J_2\ddot{\theta}_2 \tag{3.114}$$

$$J_2\ddot{\theta}_2 = k_T(\theta_1 - \theta_2) + r_2F_M \tag{3.115}$$

Hoisted mass:

$$\sum F_z = m\ddot{z} \tag{3.116}$$

$$m\ddot{z} = -b\dot{z} + k(r_1\theta_1 - z) - W \tag{3.117}$$

The foregoing three second-order equations of motion can be combined into a single sixth-order equation or written as six first-order equations. Selecting the latter in order to prepare for digital simulation of the system response, we define the following state variables:

$$x_1 = \theta_1,\ \ x_2 = \dot{\theta}_1,\ \ x_3 = \theta_2,\ \ x_4 = \dot{\theta}_2,\ \ x_5 = z,\ \ x_6 = \dot{z} \tag{3.118}$$

With these definitions, the state space representation of the system is given by the following six equations:

$$\dot{x}_1 = x_2 \tag{3.119}$$

$$\dot{x}_2 = [k_T(x_3 - x_1) - r_1k(r_1x_1 - x_5)]/J_1 \tag{3.120}$$

$$\dot{x}_3 = x_4 \tag{3.121}$$

$$\dot{x}_4 = [r_2F_M - k_T(x_3 - x_1)]/J_2 \tag{3.122}$$

$$\dot{x}_5 = x_6 \tag{3.123}$$

$$\dot{x}_6 = [-W - bx_6 + k(r_1x_1 - x_5)]/m \tag{3.124}$$