1.3.3 The Smallest Scales of Turbulence

As stated in the preceding subsection, we regard turbulence as a continuum phenomenon because the smallest scales of turbulence are much larger than any molecular length scale. We can estimate the magnitude of the smallest scales by appealing to dimensional analysis, and thereby confirm this claim. Of course, to establish the relevant dimensional quantities, we must first consider the physics of turbulence at very small length scales.

We begin by noting that the cascade process present in all turbulent flows involves a transfer of **turbulence kinetic energy** (per unit mass), k, from larger eddies to smaller eddies. Dissipation of kinetic energy to heat through the action of molecular viscosity occurs at the scale of the smallest eddies. Because small-scale motion tends to occur on a short time scale, we can reasonably assume that such motion is independent of the relatively slow dynamics of the large eddies and of the mean flow. Hence, the smaller eddies should be in a state where the rate of receiving energy from the larger eddies is very nearly equal to the rate at which the smallest eddies dissipate the energy to heat. This is one of the premises of Kolmogorov's (1941) **universal equilibrium theory**. Hence, the motion at the smallest scales should depend only upon: (a) the rate at which the larger eddies supply energy, $\epsilon = -dk/dt$, and (b) the kinematic viscosity, ν.

Having established ϵ (whose dimensions are length^2/time^3) and ν (whose dimensions are length^2/time) as the appropriate dimensional quantities, it is a simple matter to form the following length (η), time (τ) and velocity (υ) scales.

$$\eta \equiv \left(\nu^3/\epsilon\right)^{1/4}, \qquad \tau \equiv \left(\nu/\epsilon\right)^{1/2}, \qquad \upsilon \equiv \left(\nu\epsilon\right)^{1/4} \tag{1.1}$$

These are the **Kolmogorov scales** of length, time and velocity.

Figure 1.5: *Andrei Nikolaevich Kolmogorov (1903-1987), whose classic 1941 paper on the universal equilibrium theory of turbulence provided an early foundation for an understanding of turbulent fluid motion.*

To appreciate how small the Kolmogorov length scale is, for example, estimates based on properties of typical turbulent boundary layers indicate the following. For an automobile moving at 65 mph, the Kolmogorov length scale near the driver's window is about $\eta \approx 1.8 \cdot 10^{-4}$ inch. Also, on a day when the temperature is 68° F, the mean free path of air, i.e., the average distance traveled by a molecule between collisions, is $\ell_{mfp} \approx 2.5 \cdot 10^{-6}$ inch. Therefore,

$$\frac{\eta}{\ell_{mfp}} \approx 72 \tag{1.2}$$

so that the Kolmogorov length is indeed much larger than the mean free path of air, which, in turn, is typically 10 times the molecular diameter.

1.3.4 Spectral Representation and the Kolmogorov -5/3 Law

To provide further insight into the description of turbulence presented above, it is worthwhile to cast the discussion in a bit more quantitative form. Since turbulence contains a continuous spectrum of scales, it is often convenient to do our analysis in terms of the **spectral distribution** of energy. In general, a spectral representation is a Fourier decomposition into wavenumbers, κ, or, equivalently, wavelengths, $\lambda = 2\pi/\kappa$. While this text, by design, makes only modest use of Fourier-transform methods, there are a few interesting observations we can make now without considering all of the complexities involved in the mathematics of Fourier transforms. In the present context, we think of the reciprocal of κ as the eddy size.

If $E(\kappa)d\kappa$ is the turbulence kinetic energy contained between wavenumbers κ and $\kappa + d\kappa$, we can say

$$k = \int_0^\infty E(\kappa)\, d\kappa \tag{1.3}$$

Recall that k is the kinetic energy per unit mass of the fluctuating turbulent velocity. Correspondingly, the **energy spectral density** or **energy spectrum function**, $E(\kappa)$, is related to the Fourier transform of k.

Observing that turbulence is so strongly driven by the large eddies, we expect $E(\kappa)$ to be a function of a length characteristic of the larger eddies, ℓ, and the mean strain rate, S, which feeds the turbulence through direct interaction of the mean flow and the large eddies. Additionally, since turbulence is always dissipative, we expect $E(\kappa)$ to depend upon ν and ϵ. By definition, it also must depend upon κ. For high Reynolds number turbulence, dimensional analysis suggests, and measurements confirm, that k can be expressed in terms of ϵ and ℓ according to [Taylor (1935)]

$$\epsilon \sim \frac{k^{3/2}}{\ell} \quad \Longrightarrow \quad k \sim (\epsilon \ell)^{2/3} \tag{1.4}$$

Although we have not yet quantified the length scale ℓ, it is the primary length scale most turbulence models are based on. In our discussion of two-point correlations in Chapter 2, an alternative to the spectral representation of turbulence, we will find that one measure of ℓ is known as the **integral length scale**. In most turbulence-modeling analysis, we assume there is a wide separation of scales, which means we implicitly assume ℓ is very large compared to the Kolmogorov length scale, viz.,

$$\ell \gg \eta \tag{1.5}$$

Substituting the estimate of ϵ from Equation (1.4) into the Kolmogorov length scale, we find

$$\frac{\ell}{\eta} = \frac{\ell}{\left(\nu^3/\epsilon\right)^{1/4}} \sim \frac{\ell\left(k^{3/2}/\ell\right)^{1/4}}{\nu^{3/4}} \sim Re_T^{3/4} \qquad \text{where} \qquad Re_T \equiv \frac{k^{1/2}\ell}{\nu} \tag{1.6}$$

The quantity Re_T is the **turbulence Reynolds number**. It is based on the velocity characteristic of the turbulent motions as represented by the square root of k, the turbulence length scale, ℓ, and the kinematic viscosity of the fluid, ν. Thus, the condition $\ell \gg \eta$ holds provided we have high Reynolds number turbulence in the sense that

$$Re_T \gg 1 \tag{1.7}$$

The existence of a wide separation of scales is a central assumption Kolmogorov made as part of his universal equilibrium theory. That is, he hypothesized that for very large Reynolds number, there is a range of eddy sizes between the largest and smallest for which the cascade process is independent of the statistics of the energy-containing eddies (so that S and ℓ can be ignored) and of the direct effects of molecular viscosity (so that ν can be ignored). The idea is that a range of wavenumbers exists in which the energy transferred by inertial effects dominates, wherefore $E(\kappa)$ depends only upon ϵ and κ. On dimensional grounds, he thus concluded that

$$E(\kappa) = C_K \epsilon^{2/3} \kappa^{-5/3}, \qquad \frac{1}{\ell} \ll \kappa \ll \frac{1}{\eta} \tag{1.8}$$

where C_K is the **Kolmogorov constant**. Because inertial transfer of energy dominates, Kolmogorov identified this range of wavenumbers as the **inertial subrange**. The existence of the inertial subrange has been verified by many experiments and numerical simulations, although many years passed before definitive data were available to confirm its existence. Figure 1.6 shows a typical energy spectrum for a turbulent flow.

While Equation (1.8) is indeed consistent with measurements, it is not the only form that can be deduced from dimensional analysis. Unfortunately, this

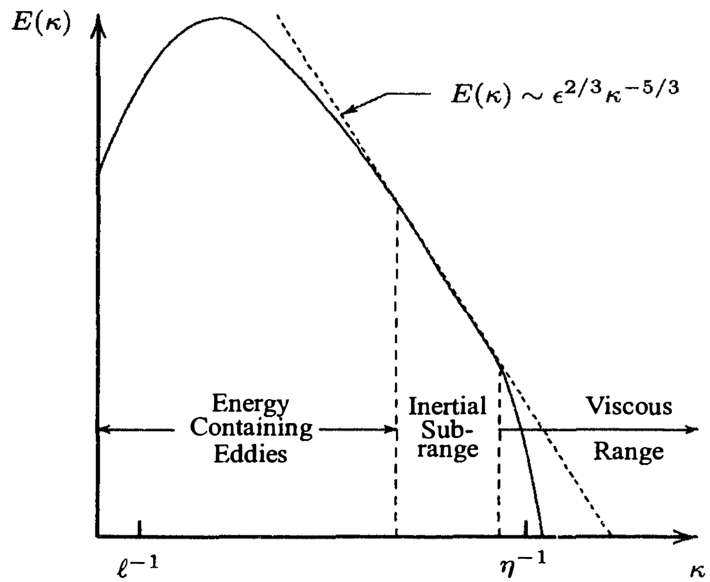

Figure 1.6: *Energy spectrum for a turbulent flow—log-log scales.*

is one of the shortcomings of dimensional analysis, i.e., the results we obtain are rarely unique. For example, lacking Kolmogorov's physical intuition, some researchers would retain ν as a dimensional quantity upon which $E(\kappa)$ depends as well as ϵ and κ. Then, a perfectly valid alternative to Equation (1.8) is

$$E(\kappa) = \epsilon^{1/4}\nu^{5/4} f(\kappa\eta), \qquad \eta = (\nu^3/\epsilon)^{1/4} \tag{1.9}$$

where $f(\kappa\eta)$ is an undetermined function. This form reveals nothing regarding the variation of $E(\kappa)$ with κ, which is a straightforward illustration of how dimensional analysis, although helpful, is insufficient to deduce physical laws.

Afzal and Narasimha (1976) use the more-powerful concepts from perturbation theory (Appendix B) to remove this ambiguity and determine the asymptotic variation of the function f in the inertial subrange. In their analysis, they assume that for small scales, corresponding to large wavenumbers, the energy spectrum function is given by Equation (1.9). This represents the *inner solution.*

Afzal and Narasimha also assume that viscous effects are unimportant for the largest eddies, and that if the only relevant scales are k and ℓ, the energy spectrum function is given by

$$E(\kappa) = k\ell g(\kappa\ell) \tag{1.10}$$

where k is the turbulence kinetic energy, ℓ is the large-eddy length scale discussed above, and $g(\kappa\ell)$ is a second undetermined function. Although we omit the details here for the sake of brevity, we can exclude explicit dependence of $E(\kappa)$ on strain rate, S, since it is proportional to $k^{1/2}/\ell$ for high Reynolds number boundary layers. This represents the *outer solution.*

Finally, they *match* the two solutions, which means they insist that the inner and outer solutions are identical when $\kappa\eta$ is small and $\kappa\ell$ is large, i.e.,

$$\epsilon^{1/4}\nu^{5/4}f(\kappa\eta) = k\ell g(\kappa\ell) \qquad \text{for} \qquad \kappa\eta \ll 1 \;\; \text{and} \;\; \kappa\ell \gg 1 \tag{1.11}$$

In words, this matching operation assumes that

> *"Between the viscous and the energetic scales in any turbulent flow exists an overlap domain over which the solutions [characterizing] the flow in the two corresponding limits must match as Reynolds number tends to infinity."*

The qualification regarding Reynolds number means it must be large enough to permit a wide separation of scales so that $\ell \gg \eta$. To complete the matching operation, Afzal and Narasimha proceed as follows. In the spirit of singular-perturbation theory, the matching operation presumes that the *functional forms* of the inner and outer solutions are the same in the overlap region. This is a much stronger condition than requiring the two solutions to have the same value at a given point. Hence, if their functional forms are the same, so are their first derivatives. Differentiating both sides of Equation (1.11) with respect to κ gives

$$\eta\epsilon^{1/4}\nu^{5/4}f'(\kappa\eta) = k\ell^2 g'(\kappa\ell) \qquad \text{for} \qquad \kappa\eta \ll 1 \;\; \text{and} \;\; \kappa\ell \gg 1 \tag{1.12}$$

Then, noting that the Kolmogorov length scale is $\eta = \nu^{3/4}\epsilon^{-1/4}$ while Equation (1.4) tells us $k = \epsilon^{2/3}\ell^{2/3}$, we can rewrite Equation (1.12) as

$$\nu^2 f'(\kappa\eta) = \epsilon^{2/3}\ell^{8/3}g'(\kappa\ell) \qquad \text{for} \qquad \kappa\eta \ll 1 \;\; \text{and} \;\; \kappa\ell \gg 1 \tag{1.13}$$

Finally, multiplying through by $\kappa^{8/3}\epsilon^{-2/3}$ and using the fact that $\nu^2\epsilon^{-2/3} = \eta^{8/3}$, we arrive at the following equation.

$$(\kappa\eta)^{8/3}f'(\kappa\eta) = (\kappa\ell)^{8/3}g'(\kappa\ell) \qquad \text{for} \qquad \kappa\eta \ll 1 \;\; \text{and} \;\; \kappa\ell \gg 1 \tag{1.14}$$

If there is a wide separation of scales, we can regard $\kappa\eta$ and $\kappa\ell$ as separate independent variables. Thus, Equation (1.14) says that a function of one independent variable, $\kappa\eta$, is equal to a function of a different independent variable, $\kappa\ell$. This can be true only if both functions tend to a constant value in the indicated limits. Thus, in the Afzal-Narasimha overlap domain, which is the inertial subrange,

$$(\kappa\eta)^{8/3}f'(\kappa\eta) = \text{constant} \quad \Longrightarrow \quad f(\kappa\eta) = C_K(\kappa\eta)^{-5/3} \tag{1.15}$$

where C_K is a constant. Combining Equations (1.9) and (1.15), we again arrive at the Kolmogorov inertial-subrange relation, viz.,

$$E(\kappa) = C_K\epsilon^{2/3}\kappa^{-5/3} \tag{1.16}$$

which is identical to Equation (1.8).

Although the Kolmogorov $-5/3$ law is of minimal use in conventional turbulence models, it is of central importance in work on Direct Numerical Simulation (DNS), Large Eddy Simulation (LES), and Detached Eddy Simulation (DES), which we discuss in Chapter 8. The Kolmogorov $-5/3$ law is so well established that, as noted by Rogallo and Moin (1984), theoretical or numerical predictions are regarded with skepticism if they fail to reproduce it. Its standing is as important as the law of the wall, which we discuss in the next subsection.

1.3.5 The Law of the Wall

The **law of the wall** is one of the most famous empirically-determined relationships in turbulent flows near solid boundaries. Measurements show that, for both internal and external flows, the streamwise velocity in the flow near the wall varies logarithmically with distance from the surface. This behavior is known as the law of the wall. In this section, we use both dimensional analysis and matching arguments to infer this logarithmic variation.

Observation of high Reynolds number turbulent boundary layers reveals a useful, approximate description of the near-surface turbulence statistics. We find that effects of the fluid's inertia and the pressure gradient are small near the surface. Consequently, the statistics of the flow near the surface in a turbulent boundary layer are established by two primary mechanisms. The first is the rate at which momentum is transferred to the surface, per unit area per unit time, which is equal to the local shear stress, τ. The second is molecular diffusion of momentum, which plays an important role very close to the surface. Observations also indicate that the details of the eddies farther from the surface are of little importance to the near-wall flow statistics.

The validity of this approximate description improves with decreasing y/δ, where δ is the boundary-layer thickness. This is true because the ratio of typical eddy size far from the surface to eddy size close to the surface increases as y/δ decreases. In other words, since δ increases with Reynolds number, we find a wide separation of scales at high Reynolds numbers. The astute reader will note interesting parallels between this description of the turbulent boundary layer and the general description of turbulence presented in Subsection 1.3.2. Note, however, that the analogy is mathematical rather than physical. This analogy is discussed, for example, by Mellor (1972) and by Afzal and Narasimha (1976).

Although τ varies near the surface, the variation with distance from the surface, y, is fairly slow. Hence, for the dimensional-analysis arguments to follow, we can use the surface shear stress, τ_w, in place of the local shear stress. Also, we denote the molecular viscosity of the fluid by μ. Since turbulence behaves the same in gases as in liquids, it is reasonable to begin with τ_w/ρ and kinematic viscosity, $\nu = \mu/\rho$, as our primary dimensional quantities, effectively eliminating fluid density, ρ, as a primary dimensional quantity.