Although the Kolmogorov $-5/3$ law is of minimal use in conventional turbulence models, it is of central importance in work on Direct Numerical Simulation (DNS), Large Eddy Simulation (LES), and Detached Eddy Simulation (DES), which we discuss in Chapter 8. The Kolmogorov $-5/3$ law is so well established that, as noted by Rogallo and Moin (1984), theoretical or numerical predictions are regarded with skepticism if they fail to reproduce it. Its standing is as important as the law of the wall, which we discuss in the next subsection.

1.3.5 The Law of the Wall

The **law of the wall** is one of the most famous empirically-determined relationships in turbulent flows near solid boundaries. Measurements show that, for both internal and external flows, the streamwise velocity in the flow near the wall varies logarithmically with distance from the surface. This behavior is known as the law of the wall. In this section, we use both dimensional analysis and matching arguments to infer this logarithmic variation.

Observation of high Reynolds number turbulent boundary layers reveals a useful, approximate description of the near-surface turbulence statistics. We find that effects of the fluid's inertia and the pressure gradient are small near the surface. Consequently, the statistics of the flow near the surface in a turbulent boundary layer are established by two primary mechanisms. The first is the rate at which momentum is transferred to the surface, per unit area per unit time, which is equal to the local shear stress, τ. The second is molecular diffusion of momentum, which plays an important role very close to the surface. Observations also indicate that the details of the eddies farther from the surface are of little importance to the near-wall flow statistics.

The validity of this approximate description improves with decreasing y/δ, where δ is the boundary-layer thickness. This is true because the ratio of typical eddy size far from the surface to eddy size close to the surface increases as y/δ decreases. In other words, since δ increases with Reynolds number, we find a wide separation of scales at high Reynolds numbers. The astute reader will note interesting parallels between this description of the turbulent boundary layer and the general description of turbulence presented in Subsection 1.3.2. Note, however, that the analogy is mathematical rather than physical. This analogy is discussed, for example, by Mellor (1972) and by Afzal and Narasimha (1976).

Although τ varies near the surface, the variation with distance from the surface, y, is fairly slow. Hence, for the dimensional-analysis arguments to follow, we can use the surface shear stress, τ_w, in place of the local shear stress. Also, we denote the molecular viscosity of the fluid by μ. Since turbulence behaves the same in gases as in liquids, it is reasonable to begin with τ_w/ρ and kinematic viscosity, $\nu = \mu/\rho$, as our primary dimensional quantities, effectively eliminating fluid density, ρ, as a primary dimensional quantity.

Since the dimensions of the quantity τ_w/ρ are length²/time², while those of ν are length²/time, clearly we can derive a velocity scale, u_τ, defined by

$$u_\tau \equiv \sqrt{\frac{\tau_w}{\rho}} \tag{1.17}$$

and a length scale, ν/u_τ. The quantity u_τ is known as the **friction velocity**, and is a velocity scale representative of velocities close to a solid boundary. If we now postulate that the mean velocity gradient, $\partial U/\partial y$, can be correlated as a function of u_τ, ν/u_τ and y, dimensional analysis yields

$$\frac{\partial U}{\partial y} = \frac{u_\tau}{y} F\left(u_\tau y/\nu\right) \tag{1.18}$$

where $F(u_\tau y/\nu)$ is presumed to be a universal function. Examination of experimental data for a wide range of boundary layers [see, for example, Coles and Hirst (1969)], indicates that, as a good leading-order approximation,

$$F\left(u_\tau y/\nu\right) \to \frac{1}{\kappa} \qquad \text{as} \qquad u_\tau y/\nu \to \infty \tag{1.19}$$

where κ is **Kármán's constant**. The function $F(u_\tau y/\nu)$ approaching a constant value is consistent with the notion that viscous effects cease to matter far from the surface, i.e., if it varies with $u_\tau y/\nu$ it would thus depend upon ν. Integrating over y, we arrive at the famous law of the wall, viz.,

$$\frac{U}{u_\tau} = \frac{1}{\kappa} \ell n \frac{u_\tau y}{\nu} + C \tag{1.20}$$

where C is a dimensionless integration constant. Correlation of measurements indicate $C \approx 5.0$ for smooth surfaces and $\kappa \approx 0.41$ for smooth and rough surfaces [see Kline et al. (1969)].

Figure 1.7 shows a typical velocity profile for a turbulent boundary layer. The graph displays the dimensionless velocity, u^+, and distance, y^+, defined as:

$$u^+ \equiv \frac{U}{u_\tau} \quad \text{and} \quad y^+ \equiv \frac{u_\tau y}{\nu} \tag{1.21}$$

The velocity profile matches the law of the wall for values of y^+ in excess of about 30. As Reynolds number increases, the maximum value of y^+ at which the law of the wall closely matches the actual velocity increases.

Observe that three distinct regions are discernible, viz., the **viscous sublayer**, the **log layer** and the **defect layer**. By definition, the log layer is the portion of the boundary layer where the sublayer and defect layer merge and the law of the wall accurately represents the velocity. It is not a distinct layer. Rather, it is

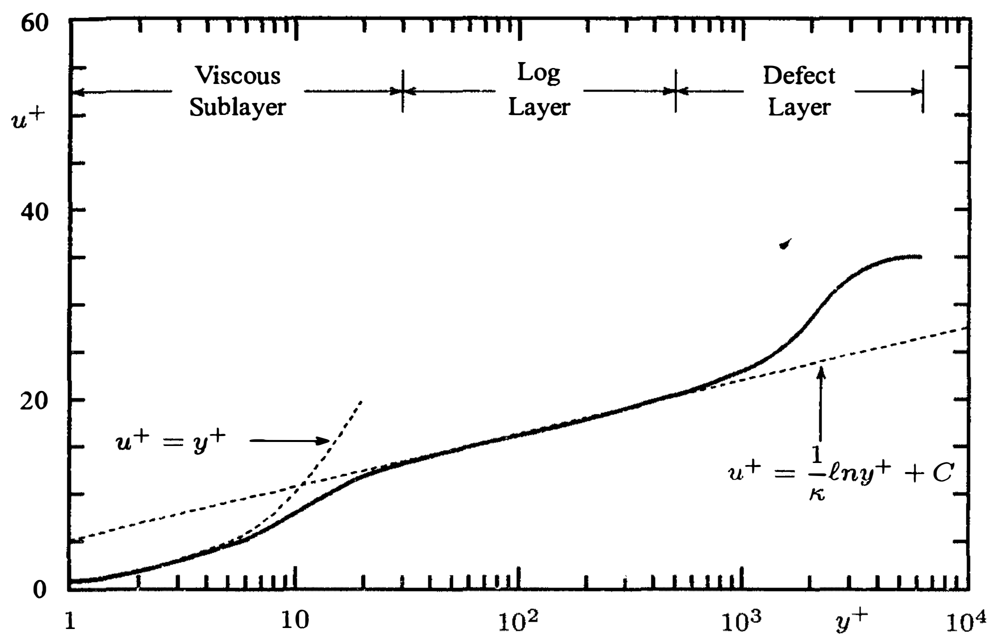

Figure 1.7: *Typical velocity profile for a turbulent boundary layer.*

an overlap region between the inner and outer parts of the boundary layer. As we will see in the following discussion, originally presented by Millikan (1938), it is an overlap domain similar to that of the Afzal-Narasimha analysis of the preceding subsection.

Assuming the velocity in the viscous sublayer should depend only upon u_τ, ν and y, we expect to have a relationship of the form

$$U = u_\tau f\left(y^+\right) \tag{1.22}$$

where $f(y^+)$ is a dimensionless function. This general functional form is often referred to as the **law of the wall**, and Equation (1.20) is simply a more explicit form. By contrast, in the defect layer, numerous experimenters including Darcy, von Kármán and Clauser found that velocity data correlate reasonably well with the so-called **velocity-defect law** or **Clauser defect law**:

$$U = U_e - u_\tau g(\eta), \qquad \eta \equiv \frac{y}{\Delta} \tag{1.23}$$

where U_e is the velocity at the boundary-layer edge and $g(\eta)$ is another dimensionless function. The quantity Δ is a thickness characteristic of the outer portion of the boundary layer.

Hence, we have an inner length scale ν/u_τ and an outer length scale Δ. Millikan's postulate is that if a wide separation of scales exists in the sense that

$$\frac{\nu}{u_\tau} \ll \Delta \tag{1.24}$$

then an overlap domain exists such that

$$u_\tau f\left(y^+\right) = U_e - u_\tau g(\eta) \qquad \text{for} \qquad y^+ \gg 1 \text{ and } \eta \ll 1 \tag{1.25}$$

We can complete the matching without explicit knowledge of the functions f and g by differentiating Equation (1.25) with respect to y. Hence,

$$\frac{u_\tau^2}{\nu} f'\left(y^+\right) = -\frac{u_\tau}{\Delta} g'(\eta) \qquad \text{for} \qquad y^+ \gg 1 \text{ and } \eta \ll 1 \tag{1.26}$$

Then, multiplying through by y/u_τ, we find

$$y^+ f'\left(y^+\right) = -\eta g'(\eta) \qquad \text{for} \qquad y^+ \gg 1 \text{ and } \eta \ll 1 \tag{1.27}$$

Thus, since a wide separation of scales means we can regard y^+ and η as independent variables, clearly the only way a function of y^+ can be equal to a function of η is for both to be equal to a constant. Therefore,

$$y^+ f'\left(y^+\right) = \text{constant} = \frac{1}{\kappa} \quad \Longrightarrow \quad f(y^+) = \frac{1}{\kappa}\ell n y^+ + C \tag{1.28}$$

which, when combined with Equation (1.22), yields Equation (1.20).

As noted earlier, the value of C for a perfectly-smooth surface is $C \approx 5.0$. For surfaces with roughness elements of average height k_s, the law of the wall still holds, although C is a function of k_s. Figure 1.8 illustrates how C varies as a function of the dimensionless roughness height given by

$$k_s^+ \equiv \frac{u_\tau k_s}{\nu} \tag{1.29}$$

As shown, as k_s increases, the value of C decreases. For large roughness height, measurements of Nikuradse [Schlichting-Gersten (1999)] show that

$$C \rightarrow 8.0 - \frac{1}{\kappa}\ell n k_s^+, \qquad k_s^+ \gg 1 \tag{1.30}$$

Substituting this value of C into the law of the wall as represented in Equation (1.20) yields:

$$\frac{U}{u_\tau} = \frac{1}{\kappa}\ell n\left(\frac{y}{k_s}\right) + 8.0 \qquad \text{(completely-rough wall)} \tag{1.31}$$

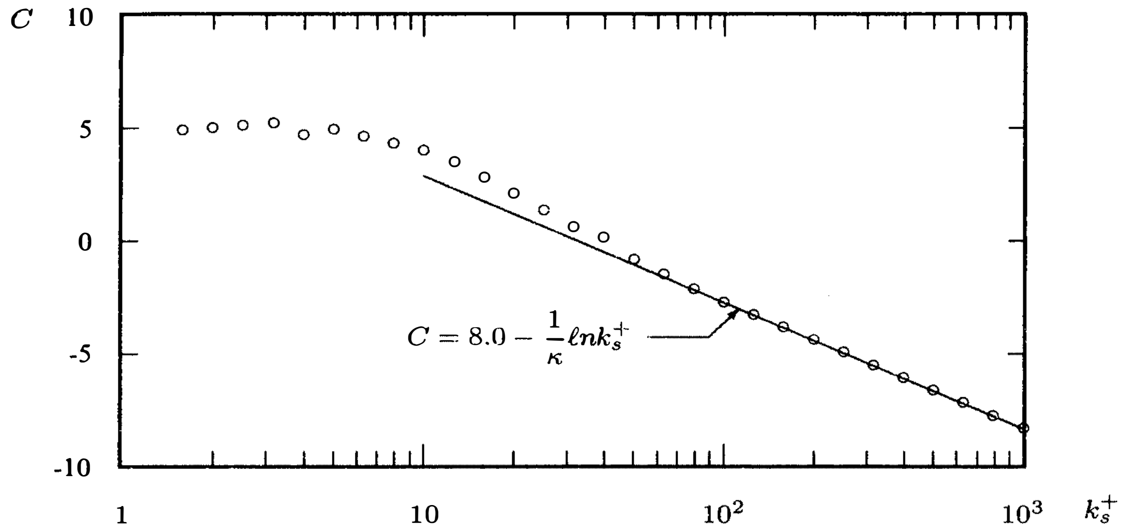

Figure 1.8: *Constant in the law of the wall, C, as a function of surface roughness;* o *based on measurements of Nikuradse [Schlichting-Gersten (1999)].*

The absence of viscosity in this equation is consistent with the notion that the surface "shear stress" is due to pressure drag on the roughness elements.

The defect layer lies between the log layer and the edge of the boundary layer. The velocity asymptotes to the law of the wall as $y/\delta \to 0$, and makes a noticeable departure from logarithmic behavior approaching the freestream. Again, from correlation of measurements, the velocity behaves as

$$U^+ = \frac{1}{\kappa}\ell n y^+ + C + \frac{2\Pi}{\kappa}\sin^2\left(\frac{\pi}{2}\frac{y}{\delta}\right) \tag{1.32}$$

where Π is **Coles' wake-strength parameter** [Coles and Hirst (1969)] and δ is boundary-layer thickness. It varies with pressure gradient, and for constant pressure, correlation of measurements suggests $\Pi \approx 0.6$. Equation (1.32) is often referred to as the composite law of the wall and **law of the wake** profile.

As demonstrated by Clauser (1956) experimentally and justified with perturbation methods by others analytically [see, for example, Kevorkian and Cole (1981), Van Dyke (1975) or Wilcox (1995a)], the velocity in the defect layer varies in a self-similar manner provided the **equilibrium parameter** defined by

$$\beta_T \equiv \frac{\delta^*}{\tau_w}\frac{dP}{dx} \tag{1.33}$$

is constant. The quantities δ^* and P in Equation (1.33) are displacement thickness and mean pressure, respectively. As demonstrated by Wilcox (1993b), even when β_T is not constant, if it is not changing too rapidly, the value for Π is close

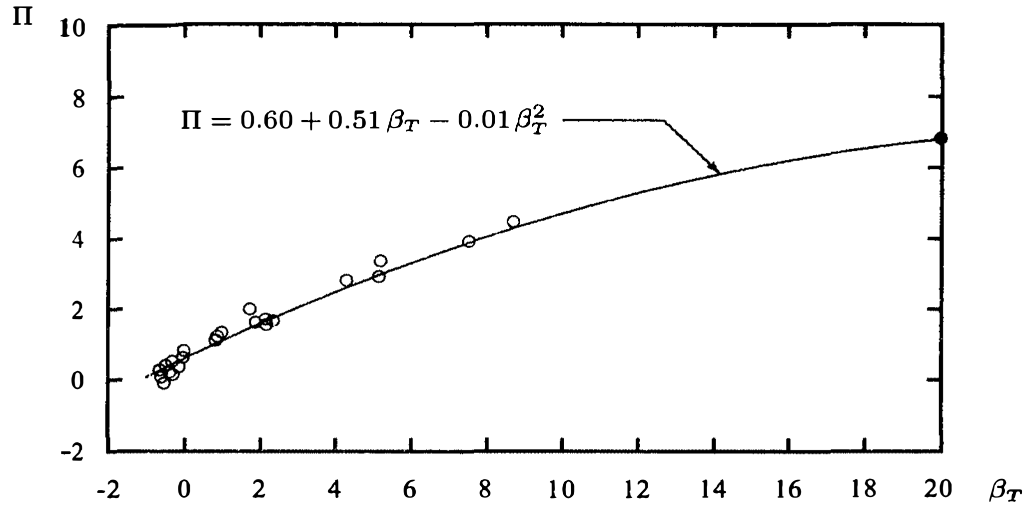

Figure 1.9: *Coles' wake-strength parameter, Π, as a function of pressure gradient; ○ from data of Coles and Hirst (1969); • Skare and Krogstad (1994).*

to the value corresponding to the local value of β_T. Figure 1.9 shows how Π varies with pressure gradient for the so-called **equilibrium turbulent boundary layer**, i.e., a boundary layer for which β_T is constant.

1.3.6 Power Laws

Often, as an approximation, turbulent boundary-layer profiles are represented by a **power-law** relationship. That is, we sometimes say

$$\frac{U}{U_e} = \left(\frac{y}{\delta}\right)^{1/n} \tag{1.34}$$

where n is typically an integer between 6 and 8. A value of $n = 7$, first suggested by Prandtl [Schlichting-Gersten (1999)], yields a good approximation at high Reynolds number for the flat-plate boundary layer. Figure 1.10 compares a 1/7 power-law profile with measurements. The agreement between measured values for a plate-length Reynolds number of $Re_x = 1.09 \cdot 10^7$ and the approximate profile is surprisingly good with differences everywhere less than 3%.

Recently, Barenblatt and others [see, for example, Barenblatt (1991), George, Knecht and Castillo (1992), Barenblatt (1993) and Barenblatt, Chorin and Prostokishin (1997)] have challenged the validity of the law of the wall. Their contention is that a power-law variation of the velocity in the inner layer better correlates pipe-flow measurements and represents a more realistic description of the turbulence in a boundary layer.

The critical assumption that Barenblatt et al. challenge is the existence of a wide separation of scales, i.e., large $\delta/(\nu/u_\tau)$. They maintain that the turbulence

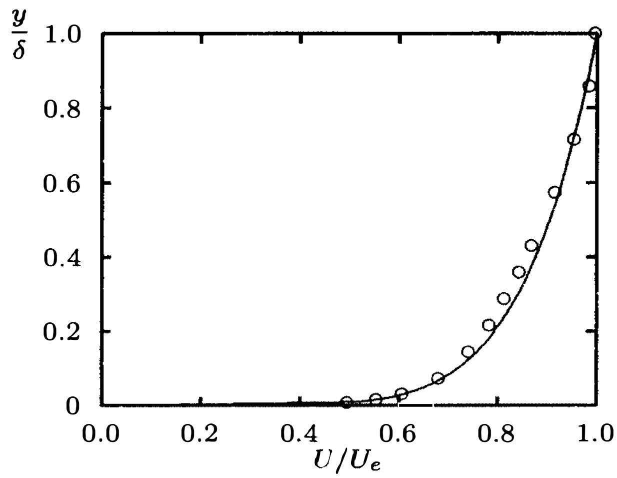

Figure 1.10: *Power-law velocity profile;* — $U/U_e = (y/\delta)^{1/7}$*;* ∘ *Wieghardt data at* $Re_x = 1.09 \cdot 10^7$ *[Coles and Hirst (1969)].*

in the overlap region is Reynolds-number dependent. If this is true, the law of the wall and defect-law Equations (1.22) and (1.23), respectively, must be replaced by

$$U = u_\tau \tilde{f}\left(y^+, Re\right) \qquad \text{and} \qquad U = U_e - u_o \tilde{g}(\eta, Re) \tag{1.35}$$

where Re is an appropriate Reynolds number, $\tilde{f}$ and $\tilde{g}$ are universal functions, and u_o is a velocity scale that is not necessarily equal to u_τ. Equivalently, the Barenblatt et al. hypothesis replaces Equation (1.18) by

$$\frac{\partial U}{\partial y} = \frac{u_\tau}{y} \Phi\left(y^+, Re\right) \tag{1.36}$$

where the universal function $\Phi(y^+, Re)$ appears in place of $F(y^+)$.

In the Millikan argument, the assumption of a wide separation of scales implies that the boundary layer possesses self-similar solutions both in the defect layer and the sublayer, in the sense that a similarity variable, e.g., $y^+ = u_\tau y/\nu$ and $\eta = y/\Delta$, exists in each region. The assumption that we can regard y^+ and η as distinct independent variables in the overlap region is described as a condition of **complete similarity**. By contrast, the Barenblatt hypothesis corresponds to **incomplete similarity**. Barenblatt (1979) discusses the distinction between complete and incomplete similarity in detail.

Under the assumption of incomplete similarity, there is no a priori reason for the function $\Phi(y^+, Re)$ to approach a constant value in the limit $y^+ \to \infty$, even when $Re \to \infty$. Rather, Barenblatt et al. argue that for large y^+,

$$\Phi\left(y^+, Re\right) = A\left(y^+\right)^\alpha \tag{1.37}$$

where the coefficient A and the exponent α are presumed to be functions of Reynolds number. In the nomenclature of Barenblatt, Chorin and Prostokishin

(1997), they assume "incomplete similarity in the parameter $[y^+]$ and no similarity in the parameter Re." Combining Equations (1.36) and (1.37) yields

$$\frac{\partial U^+}{\partial y^+} = A\left(y^+\right)^{\alpha-1} \quad \Longrightarrow \quad U^+ = \frac{A}{\alpha}\left(y^+\right)^{\alpha} \tag{1.38}$$

Based primarily on experimental data for pipe flow gathered by Nikuradse in the 1930's [Schlichting-Gersten (1999)], Barenblatt, Chorin and Prostokishin conclude that

$$A = 0.577\ell n Re + 2.50 \qquad \text{and} \qquad \alpha = \frac{1.5}{\ell n Re} \tag{1.39}$$

where Re is Reynolds number based on average velocity and pipe diameter.

To test the Barenblatt et al. alternative to the law of the wall, Zagarola, Perry and Smits (1997) have performed an analysis based on more recent experiments by Zagarola (1996). The advantage of these data lies in the much wider range of Reynolds numbers considered, especially large values, relative to those considered by Nikuradse. They conclude that the classical law of the wall provides closer correlation with measurements than the power law given by combining Equations (1.38) and (1.39), although they recommend a somewhat larger value for κ of 0.44.

To remove the possibility that the 60-year-old data of Nikuradse provide a poor correlation of A and α, Zagarola, Perry and Smits determine their values from the Zagarola data, concluding that

$$A = 0.7053\ell n Re + 0.3055 \qquad \text{and} \qquad \alpha = \frac{1.085}{\ell n Re} + \frac{6.535}{(\ell n Re)^2} \tag{1.40}$$

Even with these presumably more-accurate values, the logarithmic law of the wall still provides closer correlation with measurements than the power-law form.

This prompted Barenblatt, Chorin and Prostokishin (1997) — with a questionable argument — to demonstrate that at high Reynolds number the Zagarola experiments have significant surface roughness. Zagarola, Perry and Smits (1997) reject this possibility in stating that "the pipe surface was shown to be smooth."

Buschmann and Gad-el-Hak (2003) have offered what may be the final chapter of the power-law saga. They have performed an extensive analysis of mean-velocity profiles to determine if the power-law or the classical law-of-the-wall formulation provides optimum correlation of measurements. Their profiles include five sets of measurements and one data set from a Direct Numerical Simulation. After a detailed statistical analysis, they conclude that "the examined data do not indicate any statistically significant preference toward either law."