Problems

2.1 Develop the time-averaged form of the equation of state for a perfect gas, $p = \rho RT$, accounting for turbulent fluctuations in the instantaneous pressure, p, density, ρ, and temperature, T.

2.2 Suppose we have a velocity field that consists of: (i) a slowly varying component $U(t) = U_0 e^{-t/\tau}$ where U_0 and τ are constants and (ii) a rapidly varying component $u' = aU_0 \cos(2\pi t/\epsilon^2\tau)$ where a and ϵ are constants with $\epsilon \ll 1$. We want to show that by choosing $T = \epsilon\tau$, the limiting process in Equation (2.9) makes sense.

(a) Compute the exact time average of $u = U + u'$.

(b) Replace T by $\epsilon\tau$ in the slowly varying part of the time average of u and let $t_f = \epsilon^2\tau$ in the fluctuating part of u to show that

$$\overline{U + u'} = U(t) + O(\epsilon)$$

where $O(\epsilon)$ denotes a quantity that goes to zero linearly with ϵ as $\epsilon \rightarrow 0$.

(c) Repeat Parts (a) and (b) for du/dt, and verify that in order for Equation (2.13) to hold, necessarily $a \ll \epsilon$.

2.3 For an imposed periodic mean flow, a standard way of decomposing flow properties is to write

$$u(\mathbf{x},t) = U(\mathbf{x}) + \hat{u}(\mathbf{x},t) + u'(\mathbf{x},t)$$

where $U(\mathbf{x})$ is the mean-value, $\hat{u}(\mathbf{x},t)$ is the organized response component due to the imposed organized unsteadiness, and $u'(\mathbf{x},t)$ is the turbulent fluctuation. $U(\mathbf{x})$ is defined as in Equation (2.5). We also use the **Phase Average** defined by

$$<u(\mathbf{x},t)> \;\equiv \lim_{N\rightarrow\infty} \frac{1}{N} \sum_{n=0}^{N-1} u(\mathbf{x}, t + n\tau)$$

where τ is the period of the imposed excitation. Then, by definition,

$$<u(\mathbf{x},t)> \;= U(\mathbf{x}) + \hat{u}(\mathbf{x},t), \quad \overline{<u(\mathbf{x},t)>} \;= U(\mathbf{x}), \quad <\hat{u}(\mathbf{x},t)> = \hat{u}(\mathbf{x},t)$$

Verify the following. Do not assume that $\hat{u}$ is sinusoidal.

(a) $\overline{\hat{u}} = 0$	(d) $<U> = U$	(g) $<\hat{u}v> = \hat{u}<v>$
(b) $\overline{u'} = 0$	(e) $<u'> = 0$	(h) $<\hat{u}v'> = 0$
(c) $\overline{\hat{u}v'} = 0$	(f) $<Uv> = U<v>$	

2.4 Compute the difference between the Reynolds average of a triple product $\lambda\delta\sigma$ and the product of the means, $\Lambda\Delta\Sigma$.

2.5 Compute the difference between the Reynolds average of a quadruple product $\phi\psi\xi\upsilon$ and the product of the means, $\Phi\Psi\Xi\Upsilon$.

2.6 For an incompressible flow, we have an imposed freestream velocity given by

$$u(x,t) = U_o(1 - ax) + U_o ax \sin 2\pi f t$$

where a is a constant of dimension 1/length, U_o is a constant reference velocity, and f is frequency. Integrating over one period, compute the average pressure gradient, dP/dx, for $f = 0$ and $f \neq 0$ in the freestream where the inviscid Euler equation holds, i.e.,

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} = -\frac{\partial p}{\partial x}$$

2.7 Consider the Reynolds-stress equation as stated in Equation (2.34).

(a) Show how Equation (2.34) follows from Equation (2.33).

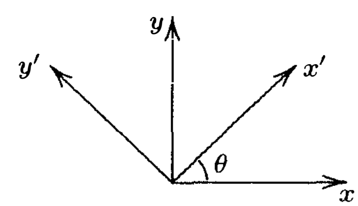
(b) Contract Equation (2.34), i.e., set $i = j$ and perform the indicated summation, to derive a differential equation for the kinetic energy of the turbulence per unit mass defined by $k \equiv \frac{1}{2}\overline{u_i' u_i'}$.

2.8 Consider the third-rank tensor $\overline{u_i' u_j' u_k'}$ appearing in Equation (2.33). In general, third-rank tensors have 27 components. Verify that this tensor has only 10 independent components and list them.

2.9 If we rotate the coordinate system about the z axis by an angle θ, the Reynolds stresses for an incompressible two-dimensional boundary layer transform according to:

$$\begin{aligned}
\tau_{xx}' &= \frac{1}{2}(\tau_{xx} + \tau_{yy}) + \frac{1}{2}(\tau_{xx} - \tau_{yy})\cos 2\theta + \tau_{xy}\sin 2\theta \\
\tau_{yy}' &= \frac{1}{2}(\tau_{xx} + \tau_{yy}) - \frac{1}{2}(\tau_{xx} - \tau_{yy})\cos 2\theta - \tau_{xy}\sin 2\theta \\
\tau_{xy}' &= \tau_{xy}\cos 2\theta - \frac{1}{2}(\tau_{xx} - \tau_{yy})\sin 2\theta \\
\tau_{zz}' &= \tau_{zz}
\end{aligned}$$

Assume the normal Reynolds stresses, $\tau_{xx} = -\overline{u'^2}$, etc. are given by Equation (2.39), and that the Reynolds shear stress is $\tau_{xy} = -\overline{u'v'} \approx \frac{3}{10}k$.



Problem 2.9

(a) Determine the angle the *principal axes* make with the xy axes, i.e., the angle that yields $\tau_{xy}' = 0$.

(b) What is the Reynolds-stress tensor, τ_{ij}', in the *principal axis* system?

2.10 Using Figure 2.6, determine the values of $\overline{u'^2}/k$, $\overline{v'^2}/k$ and $\overline{w'^2}/k$ for dimensionless distances from the surface of $y/\delta = 0.2$, 0.4 and 0.6. Determine the percentage differences between measured values and the following approximations.

(a) Equation (2.39)

(b) $\overline{u'^2} \approx k$, $\overline{v'^2} \approx \frac{2}{5}k$, $\overline{w'^2} \approx \frac{3}{5}k$

2.11 Verify that, for homogeneous-isotropic turbulence, the ratio of the *micro-time scale*, τ_E, to the Kolmogorov time scale varies linearly with the isotropic turbulence-intensity parameter, T'.