Problems

4.1 We wish to create a new two-equation turbulence model. Our first variable is turbulence kinetic energy, k, while our second variable is the "eddy acceleration," a. Assuming a has dimensions (length)/(time)^2, use dimensional arguments to deduce plausible algebraic dependencies of eddy viscosity, ν_T, turbulence kinetic energy dissipation rate, ϵ, and turbulence length scale, ℓ, upon k and a.

4.2 Starting with Equations (4.4) and (4.45), define $\epsilon = \beta^* \omega k$ and derive an "exact" ω equation.

4.3 Verify that the exact equation for the dissipation, ϵ, is given by Equation (4.45). That is, derive the equation that follows from taking the following moment of the Navier-Stokes equation:

$$\overline{2\nu \frac{\partial u_i'}{\partial x_j} \frac{\partial}{\partial x_j} \left[\mathcal{N}(u_i) \right]} = 0$$

where $\mathcal{N}(u_i)$ is the Navier-Stokes operator defined in Equation (2.26).

4.4 Derive the exact equation for the **enstrophy**, ω^2, defined by

$$\omega^2 \equiv \frac{1}{2} \overline{\omega_i' \omega_i'} \qquad \text{where} \qquad \omega_i' = \epsilon_{ijk} \partial u_k' / \partial x_j$$

That is, ω_i' is the fluctuating vorticity. **HINT:** Beginning with the Navier-Stokes equation, derive the equation for the vorticity, multiply by ω_i', and time average. The vector identity $\mathbf{u} \cdot \nabla \mathbf{u} = \nabla \left(\frac{1}{2} \mathbf{u} \cdot \mathbf{u} \right) - \mathbf{u} \times (\nabla \times \mathbf{u})$ should prove useful in deriving the vorticity equation.

4.5 Beginning with the k-ϵ model, make the formal change of variables $\epsilon = C_\mu \omega k$ and derive the implied k-ω model. Express your final results in standard k-ω model notation and determine the implied values for α, β, β^*, σ, σ^* and σ_d in terms of C_μ, $C_{\epsilon 1}$, $C_{\epsilon 2}$, σ_k and σ_ϵ.

4.6 Beginning with the k-ω model and with $\sigma = \sigma^* = 1/2$ and $\sigma_d = 0$, make the formal change of variables $\epsilon = \beta^* \omega k$ and derive the implied k-ϵ model. Express your final results in standard k-ϵ model notation and determine the implied values for C_μ, $C_{\epsilon 1}$, $C_{\epsilon 2}$, σ_k and σ_ϵ in terms of α, β, β^*, σ and σ^*. Assume $f_\beta = 1$ and omit the stress limiter.

4.7 Simplify the k-ϵ, k-$k\ell$, k-$k\tau$ and k-τ models for the log layer. Determine the value of Kármán's constant, κ, implied by the closure coefficient values quoted in Equations (4.49), (4.57), (4.63) and (4.66). Make a table of your results and include the value 0.40 for the k-ω model. **NOTE:** For all models, assume a solution of the fonn $dU/dy = u_\tau/(\kappa y)$, $k = u_\tau^2/\sqrt{C_\mu}$ and $\nu_T = \kappa u_\tau y$. Also, $C_\mu = C_D$ for the k-$k\ell$ model.

4.8 Simplify the k-ϵ, k-$k\ell$, k-$k\tau$ and k-τ models for homogeneous, isotropic turbulence. Determine the asymptotic decay rate for k as a function of the closure coefficient values quoted in Equations (4.49), (4.57), (4.63) and (4.66). Make a table of your results and include the decay rate of $t^{-1.27}$ for the k-ω model. (**NOTE:** You can ignore the $(\ell/y)^6$ contribution to C_{L2} for the k-$k\ell$ model.)

4.9 Beginning with Equations (4.83), derive the self-similar form of the k-ω model equations for the mixing layer between a fast stream moving with velocity U_1 and a slow stream with velocity U_2. Omit the stress limiter so that $\nu_T = k/\omega$.

(a) Assuming a streamfunction of the form $\psi(x, y) = U_1 x F(\eta)$, transform the momentum equation, and verify that $\mathcal{V}$ is as given in Table 4.3.

(b) Transform the equations for k and ω.

(c) State the boundary conditions on $\mathcal{U}$ and K for $|\eta| \to \infty$ and for $\mathcal{V}(0)$. Assume $k \to 0$ as $|y| \to \infty$.

(d) Verify that if $\omega \neq 0$ in the freestream, the only boundary conditions consistent with the similarity solution are:

$$W(\eta) \to \begin{cases} \dfrac{1}{\beta_o}, & \eta \to +\infty \\ \dfrac{U_1/U_2}{\beta_o}, & \eta \to -\infty \end{cases}$$

4.10 Using Programs **WAKE**, **MIXER** and **JET** (see Appendix C), determine the spreading rates for the five basic free shear flows according to the k-ω model with and without the stress limiter. Compare your results in tabular form. **HINT:** The limiter is defined in the array *climit(j)*, whose value is set in Subroutine *CALCS*.

4.11 Derive Equation (4.145).

4.12 Demonstrate the integral constraint on $U_1(\eta)$ in the defect-layer solution.

4.13 Determine the shape factor to $O(u_\tau/U_e)$ according to the defect-layer solution. Express your answer in terms of an integral involving $U_1(\eta)$.

4.14 Using Program **DEFECT** (see Appendix C), determine the variation of Coles' wake strength, Π, as a function of the equilibrium parameter, β_T, for Kok's k-ω model. Modify the program, noting that Kok's model does not use the stress limiter and its closure coefficients are $\alpha = 5/9$, $\beta = 3/40$, $\beta^* = 9/100$, $\sigma = 1/2$, $\sigma^* = 2/3$ and $\sigma_{do} = 1/2$. Compare your results to the correlation $\Pi = 0.60 + 0.51\,\beta_T - 0.01\,\beta_T^2$. Do your computations for $-0.35 \le \beta_T \le 20$. **HINT:** You can accomplish all of the required modifications in Subroutine *START* by changing the values of the closure coefficients and noting that setting *clim* equal to zero turns the stress limiter off.

4.15 Using Program **DEFECT** (see Appendix C), determine the variation of Coles' wake strength, Π, as a function of the equilibrium parameter, β_T, for the Launder-Sharma k-ϵ model with a stress limiter included. Make a graph that includes values obtained with and without a stress limiter and the correlation $\Pi = 0.60 + 0.51\,\beta_T - 0.01\,\beta_T^2$. Do your computations for $-0.35 \le \beta_T \le 20$. **HINT:** The limiter is defined in the array *climit(j)*, whose value is set in Subroutine *CALCS*. Its algebraic form is identical for the k-ω and k-ϵ models, so all you have to do is activate it for the k-ϵ model. Set the constant *clim* equal to 1 to maximize the effect of the limiter.

4.16 Consider a flow with freestream velocity U_∞ past a wavy wall whose shape is

$$y = \frac{1}{2}k_s \sin\left(\frac{2\pi x}{Nk_s}\right)$$

where k_s is the peak to valley amplitude and Nk_s is wavelength. The linearized incompressible solution is $U = U_\infty + u'$, $V = v'$ where

$$u' = \frac{\pi U}{N}\exp\left(-\frac{2\pi y}{Nk_s}\right)\sin\left(\frac{2\pi y}{Nk_s}\right), \qquad v' = \frac{\pi U}{N}\exp\left(-\frac{2\pi y}{Nk_s}\right)\cos\left(\frac{2\pi y}{Nk_s}\right)$$

Making an analogy between this linearized solution and the fluctuating velocity field in a turbulent flow, compute the specific dissipation rate, $\omega = \epsilon/(\beta^* k)$. Ignore contributions from the other fluctuating velocity component, w'.

4.17 For the k-ω model, very close to the surface and deep within the viscous sublayer, dissipation balances molecular diffusion in the ω equation. Assuming a solution of the form $\omega = \omega_w/(1 + Ay)^2$, solve this equation for $\omega = \omega_w$ at $y = 0$. Determine the limiting form of the solution as $\omega_w \to \infty$.

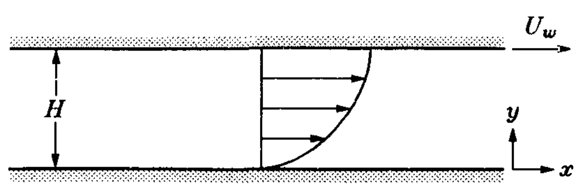
4.18 Using Program **SUBLAY** (see Appendix C), determine the variation of the constant C in the law of the wall for the k-ω model with the surface value of ω. Do your computations with (*nvisc* = 0) and without (*nvisc* = 1) viscous modifications. Let ω_w^+ assume the values 1, 3, 10, 30, 100, 300, 1000 and ∞. Be sure to use the appropriate value for input parameter *iruff*. Present your results in tabular form.

4.19 This problem studies the effect of viscous-modification closure coefficients for the k-ω model using Program **SUBLAY** (see Appendix C).

(a) Modify Subroutine *START* to permit inputting the values of R_k and R_ω (program variables *rk* and *rw*). Determine the value of R_ω that yields a smooth-wall constant in the law of the wall, C, of 5.0 for R_k = 4, 6, 8, 10 and 20.

(b) Now make provision for inputting the value of R_β (program variable *rb*). For $R_k = 6$, determine the value of R_ω that yields $C = 5.0$ when R_β = 2, 4, 8, and 12. Also, determine the maximum value of k^+ for each case.

4.20 Consider incompressible Couette flow with constant pressure, i.e., flow between two parallel plates separated by a distance H, the lower at rest and the upper moving with constant velocity U_w.



Problems 4.20 and 4.21

(a) Assuming the plates are infinite in extent, simplify the conservation of mass and momentum equations and verify that

$$(\nu + \nu_T)\frac{dU}{dy} = u_\tau^2$$

(b) Now ignore molecular viscosity. What boundary condition on U is appropriate at the lower plate?

(c) Introducing the mixing length given by

$$\ell_{mix} = \kappa y(1 - y/H)$$

solve for the velocity across the channel. **HINT:** Using partial fractions:

$$\frac{1}{y(1-y/H)} = \frac{1}{y} + \frac{1}{(H-y)}$$

Don't forget to use the boundary condition stated in Part (b).

(d) Develop a relation between friction velocity, u_τ, and the average velocity,

$$U_{avg} = \frac{1}{H}\int_0^H U(y)\, dy$$

(e) Using the k-ω model, simplify the equations for k and ω with the same assumptions made in Parts (a) and (b).

(f) Deduce the equations for k and ω that follow from changing independent variables from y to U so that

$$\nu_T \frac{d}{dy} = u_\tau^2 \frac{d}{dU}$$

(g) Assuming $k = u_\tau^2/\sqrt{\beta^*}$, simplify the equation for ω. **NOTE:** You might want to use the fact that $\sigma\sqrt{\beta^*}\kappa^2 = \beta_o - \alpha\beta^*$.

4.21 For incompressible, laminar Couette flow, we know that the velocity is given by

$$U = U_w \frac{y}{H}$$

where U_w is the velocity of the moving wall, y is distance form the stationary wall, and H is the distance between the walls.

(a) Noting that the stress limiter is inactive for laminar flow, determine the maximum Reynolds number,

$$Re_{H_c} = U_w H_c/\nu$$

at which the flow remains laminar according to the high-Reynolds-number version of the k-ω model. To arrive at your answer, you may assume that

$$\omega = \begin{cases} \dfrac{6\nu}{\beta_o y^2}, & 0 \le y \le H/2 \\ \dfrac{6\nu}{\beta_o (H-y)^2}, & H/2 \le y \le H \end{cases}$$

(b) Above what Reynolds number is ω amplified?

4.22 Using Program **PIPE** (see Appendix C), compute the skin friction for channel flow according to the Baldwin-Barth and Spalart-Allmaras models. Compare your results with the Halleen-Johnston correlation [Equation (3.139)] for $10^3 \leq Re_H \leq 10^5$. Also, compare the computed velocity profiles for $Re_H = 13750$ with the Mansour et al. DNS data, which are as follows.

$y/(H/2)$	U/U_m	$y/(H/2)$	U/U_m	$y/(H/2)$	U/U_m
0.000	0.000	0.404	0.887	0.805	0.984
0.103	0.717	0.500	0.917	0.902	0.995
0.207	0.800	0.602	0.945	1.000	1.000
0.305	0.849	0.710	0.968		

4.23 Using Program **PIPE** (see Appendix C), compute the skin friction for pipe flow according to the Baldwin-Barth and Spalart-Allmaras models. Compare your results with the Prandtl correlation [Equation (3.140)] for $10^3 \leq Re_D \leq 10^6$. Also, compare the computed velocity profiles for $Re_D = 40000$ with Laufer's data, which are as follows.

$y/(D/2)$	U/U_m	$y/(D/2)$	U/U_m	$y/(D/2)$	U/U_m
0.010	0.333	0.390	0.868	0.800	0.975
0.095	0.696	0.490	0.902	0.900	0.990
0.210	0.789	0.590	0.931	1.000	1.000
0.280	0.833	0.690	0.961		

4.24 The object of this problem is to compare predictions of one- and two-equation models with measured properties of a turbulent boundary layer with adverse ∇p. The experiment to be simulated was conducted by Schubauer and Spangenberg [see Coles and Hirst (1969) – Flow 4800]. Use Program **EDDYBL**, its menu-driven setup utility, Program **EDDYBL_DATA**, and the input data provided on the companion CD (see Appendix C). Do 3 computations using the Baldwin-Barth model, the k-ω model with viscous modifications and one of the k-ϵ models and compare computed skin friction with the following measured values.

s (ft)	c_f	s (ft)	c_f	s (ft)	c_f
2.000	$3.39 \cdot 10^{-3}$	10.333	$2.06 \cdot 10^{-3}$	17.000	$0.94 \cdot 10^{-3}$
4.500	$2.94 \cdot 10^{-3}$	13.667	$1.61 \cdot 10^{-3}$	17.833	$0.49 \cdot 10^{-3}$
7.000	$2.55 \cdot 10^{-3}$	15.333	$1.39 \cdot 10^{-3}$		

4.25 The object of this problem is to compare predictions of one- and two-equation models with measured properties of a turbulent boundary layer with adverse ∇p. The experiment to be simulated was conducted by Ludwieg and Tillman [see Coles and Hirst (1969) – Flow 1200]. Use Program **EDDYBL**, its menu-driven setup utility, Program **EDDYBL_DATA**, and the input data provided on the companion CD (see Appendix C). Do 3 computations using the k-ω, Baldwin-Barth and Jones-Launder models and compare computed skin friction with the following measured values.

s (m)	c_f	s (m)	c_f
0.782	$2.92 \cdot 10^{-3}$	2.282	$1.94 \cdot 10^{-3}$
1.282	$2.49 \cdot 10^{-3}$	2.782	$1.55 \cdot 10^{-3}$
1.782	$2.05 \cdot 10^{-3}$		

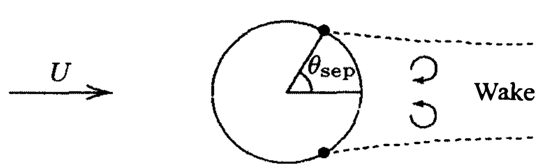
4.26 The object of this problem is to compare predictions of one- and two-equation models with measured properties of a turbulent boundary layer with adverse ∇p. The experiment to be simulated was conducted by Schubauer and Spangenberg [see Coles and Hirst (1969) – Flow 4400]. Use Program **EDDYBL**, its menu-driven setup utility, Program **EDDYBL_DATA**, and the input data provided on the companion CD (see Appendix C). Do 3 computations using the k-ω model, one of the k-ϵ models and the Spalart-Allmaras model and compare computed skin friction with the following measured values.

s (ft)	c_f	s (ft)	c_f	s (ft)	c_f
1.167	$3.40 \cdot 10^{-3}$	3.667	$2.86 \cdot 10^{-3}$	6.167	$1.33 \cdot 10^{-3}$
2.000	$3.17 \cdot 10^{-3}$	4.500	$2.38 \cdot 10^{-3}$		
2.833	$3.10 \cdot 10^{-3}$	5.333	$1.97 \cdot 10^{-3}$		

4.27 The object of this problem is to compare predictions of one- and two-equation models with measured properties of a turbulent boundary layer with adverse ∇p. The experiment to be simulated was conducted by Stratford [see Coles and Hirst (1969) – Flow 5300]. Use Program **EDDYBL**, its menu-driven setup utility, Program **EDDYBL_DATA**, and the input data provided on the companion CD (see Appendix C). Do 3 computations using the k-ω model, one of the k-ϵ models and the Spalart-Allmaras model and compare computed skin friction with the following measured values.

s (ft)	c_f	s (ft)	c_f
2.907	$3.68 \cdot 10^{-3}$	3.531	$0.55 \cdot 10^{-3}$
2.999	$2.07 \cdot 10^{-3}$	4.103	$0.53 \cdot 10^{-3}$
3.038	$0.99 \cdot 10^{-3}$		

4.28 The object of this problem is to predict the separation point for flow past a circular cylinder with the boundary-layer equations, using the measured pressure distribution. The experiment to be simulated was conducted by Patel (1968). Use Program **EDDYBL** and its menu-driven setup utility, Program **EDDYBL_DATA**, to do the computations (see Appendix C).



Problem 4.28

(a) Set freestream conditions to $p_{t_\infty} = 2147.7$ lb/ft^2, $T_{t_\infty} = 529.6^\circ$ R, $M_\infty = 0.144$ (PT1, TT1, XMA); use an initial stepsize, initial arclength and final arclength given by $\Delta s = 0.001$ ft, $s_i = 0.262$ ft and $s_f = 0.785$ ft (DS, SI, SSTOP); set the initial boundary-layer properties so that $c_f = 0.00600$, $\delta = 0.006$ ft, $H = 1.40$, $Re_\theta = 929$, (CF, DELTA, H, RETHET); set the maximum number of steps to 1000 (IEND1); and set up for $N = 47$ points to define the pressure (NUMBER). Use the following data to define the pressure distribution. The initial and final pressure gradients are zero. Use zero heat flux at the cylinder surface. Finally, set the curvature, $\mathcal{R}^{-1}$, equal to 4 ft^{-1}.

s (ft)	p_e (lb/ft^2)	s (ft)	p_e (lb/ft^2)	s (ft)	p_e (lb/ft^2)
0.0000	$2.147540 \cdot 10^3$	0.1500	$2.116199 \cdot 10^3$	0.3500	$2.055516 \cdot 10^3$
0.0025	$2.147528 \cdot 10^3$	0.1625	$2.112205 \cdot 10^3$	0.3625	$2.056591 \cdot 10^3$
0.0050	$2.147491 \cdot 10^3$	0.1750	$2.107903 \cdot 10^3$	0.3750	$2.058435 \cdot 10^3$
0.0075	$2.147429 \cdot 10^3$	0.1875	$2.103448 \cdot 10^3$	0.3875	$2.061661 \cdot 10^3$
0.0100	$2.147343 \cdot 10^3$	0.2000	$2.098378 \cdot 10^3$	0.4000	$2.066423 \cdot 10^3$
0.0125	$2.147233 \cdot 10^3$	0.2125	$2.093155 \cdot 10^3$	0.4125	$2.071954 \cdot 10^3$
0.0250	$2.146314 \cdot 10^3$	0.2250	$2.087317 \cdot 10^3$	0.4250	$2.079021 \cdot 10^3$
0.0375	$2.144796 \cdot 10^3$	0.2375	$2.081325 \cdot 10^3$	0.4375	$2.085473 \cdot 10^3$
0.0500	$2.142688 \cdot 10^3$	0.2500	$2.075334 \cdot 10^3$	0.4500	$2.089161 \cdot 10^3$
0.0625	$2.140018 \cdot 10^3$	0.2625	$2.069189 \cdot 10^3$	0.4625	$2.091004 \cdot 10^3$
0.0750	$2.136807 \cdot 10^3$	0.2750	$2.064580 \cdot 10^3$	0.4750	$2.092080 \cdot 10^3$
0.0875	$2.134021 \cdot 10^3$	0.2875	$2.060893 \cdot 10^3$	0.4875	$2.092230 \cdot 10^3$
0.1000	$2.130641 \cdot 10^3$	0.3000	$2.058588 \cdot 10^3$	0.5000	$2.092230 \cdot 10^3$
0.1125	$2.127261 \cdot 10^3$	0.3125	$2.056898 \cdot 10^3$	0.6500	$2.092230 \cdot 10^3$
0.1250	$2.123881 \cdot 10^3$	0.3250	$2.055823 \cdot 10^3$	0.7850	$2.092230 \cdot 10^3$
0.1375	$2.120194 \cdot 10^3$	0.3375	$2.055362 \cdot 10^3$		

(b) Do three computations using the low-Reynolds-number k-ω model, the Launder-Sharma k-ϵ model and the Spalart-Allmaras model. The radius of the cylinder is $R = 0.25$ ft, so that separation arclength, s_{sep}, is related to this angle by $\theta_{sep} = \pi - s_{sep}/R$.

4.29 Compute Driver and Seegmiller's $Re_H = 37500$ backstep flow using the Baldwin-Lomax algebraic model. Use Program **EDDY2C**, its menu-driven setup utility, Program **EDDY2C_DATA**, and the input data provided on the companion CD (see Appendix C).

(a) You must first run Program **EDDYBL** to establish flow properties at the upstream boundary. Modify the supplied input-data file *eddybl.dat*, using trial and error to adjust the "Maximum Arclength" (SSTOP) so that the Reynolds number based on momentum thickness is 5000.

(b) Modify the supplied input-data file *eddy2c.dat* for Program **EDDY2C** to run the computation 1000 timesteps (NEND).

(c) Make graphs of the "residual" and the value of reattachment length, x_r/H, as functions of timestep number.

(d) Discuss the value of x_r/H predicted by the Baldwin-Lomax model relative to the measured value and the values predicted by the k-ω and k-ϵ models.

NOTE: This computation will take about 30 minutes of CPU time on a 3-GHz Pentium-D microcomputer.

4.30 Compute Jovic's $Re_H = 5000$ backstep flow using the Baldwin-Lomax algebraic model. Use Program **EDDY2C**, its menu-driven setup utility, Program **EDDY2C_DATA**, and the input data provided on the companion CD (see Appendix C).

(a) You must first run Program **EDDYBL** to establish flow properties at the upstream boundary. Modify the supplied input-data file *eddybl.dat*, using trial and error to adjust the "Maximum Arclength" (SSTOP) so that the Reynolds number based on momentum thickness is 609.

(b) Modify the supplied input-data file *eddy2c.dat* for Program **EDDY2C** to run the computation 10000 timesteps (NEND).

(c) Make graphs of the "residual" and the value of reattachment length, x_r/H, as functions of timestep number.

(d) Discuss the value of x_r/H predicted by the Baldwin-Lomax model relative to the measured value and the value predicted by the k-ω model.

NOTE: This computation will take about 3 hours of CPU time on a 3-GHz Pentium-D microcomputer.